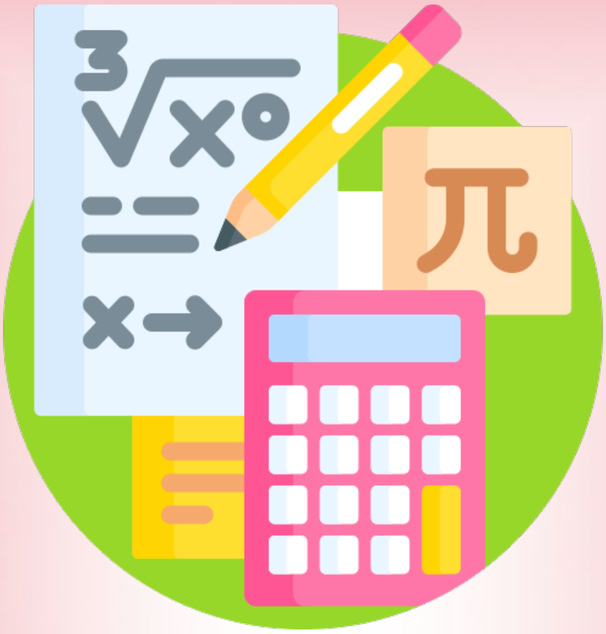
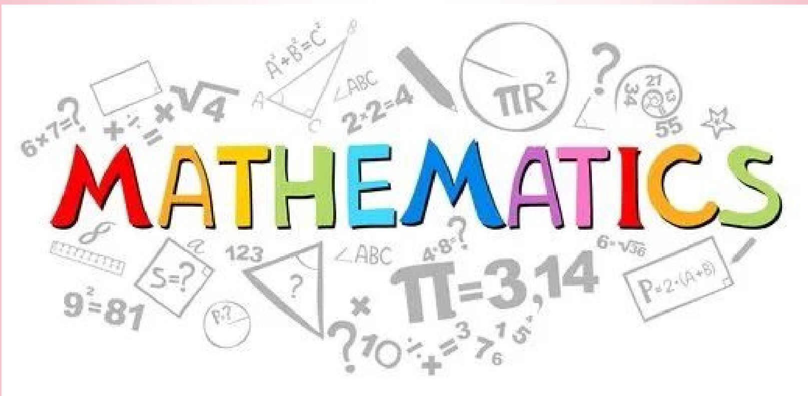
PART - 1

MATHEMATICS

READER BOOK FOR
SECONDARY COURSE





BOARD OF SCHOOL EDUCATION HUBLI, KARNATAKA

ಬೋರ್ಡ್ ಆಫ್ ಸ್ಕೂಲ್ ಎಜುಕೇಶನ್ ಹುಬ್ಬಳ್ಳಿ, ಕರ್ನಾಟಕ

08361-4061132

www.bseh.ac.in

contact@bseh.ac.in

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NUMBER SYSTEMS

From time immemorial human beings have been trying to have a count of their belongings- goods, ornaments, jewels, animals, trees, sheeps/goats, etc. by using various techniques

- putting scratches on the ground/stones

- by storing stones - one for each commodity kept/taken out.

This was the way of having a count of their belongings without having any knowledge of counting.

One of the greatest inventions in the history of civilization is the creation of numbers. You can imagine the confusion when there were no answers to questions of the type "How many?", "How much?" and the like in the absence of the knowledge of numbers. The invention of number system including zero and the rules for combining them helped people to reply questions of the type:

(i) How many apples are there in the basket?

(ii) How many speakers have been invited for addressing the meeting?

(iii) What is the number of toys on the table?

(iv) How many bags of wheat have been the yield from the field?

The answers to all these situations and many more involve the knowledge of numbers and operations on them. This points out to the need of study of number system and its extensions in the curriculum. In this lesson, we will present a brief review of natural numbers, whole numbers and integers. We shall then introduce you about rational and irrational numbers in detail. We shall end the lesson after discussing about real numbers.

OBJECTIVES

After studying this lesson, you will be able to

- *illustrate the extension of system of numbers from natural numbers to real (rationals and irrational) numbers*

- *identify different types of numbers;*
- *express an integer as a rational number;*
- *express a rational number as a terminating or non-terminating repeating decimal, and vice-versa;*
- *find rational numbers between any two rationals;*
- *represent a rational number on the number line;*
- *cites examples of irrational numbers;*
- *represent* $\sqrt{2}, \sqrt{3}, \sqrt{5}$ *on the number line;*
- *find irrational numbers betwen any two given numbers;*
- *round off rational and irrational numbers to a given number of decimal places;*
- *perform the four fundamental operations of addition, subtraction, multiplication and division on real numbers.*

1.1 EXPECTED BACKGROUND KNOWLEDGE

Basic knowledge about counting numbers and their use in day-to-day life.

1.2 RECALL OF NATURAL NUMBERS, WHOLE NUMBERS AND INTEGERS

1.2.1 Natural Numbers

Recall that the counting numbers 1, 2, 3, ... constitute the system of natural numbers. These are the numbers which we use in our day-to-day life.

Recall that there is no greatest natural number, for if 1 is added to any natural number, we get the next higher natural number, called its successor.

We have also studied about four-fundamental operations on natural numbers. For, example,

$4 + 2 = 6$, again a natural number;

$6 + 21 = 27$, again a natural number;

$22 - 6 = 16$, again a natural number, but

$2 - 6$ is not defined in natural numbers.

Similarly, $4 \times 3 = 12$, again a natural number

$12 \times 3 = 36$, again a natural number

$\frac{12}{2} = 6$ is a natural number but $\frac{6}{4}$ is not defined in natural numbers. Thus, we can say that

i) a) addition and multiplication of natural numbers again yield a natural number but

b) subtraction and division of two natural numbers may or may not yield a natural number

ii) The natural numbers can be represented on a number line as shown below.

iii) Two natural numbers can be added and multiplied in any order and the result obtained is always same. This does not hold for subtraction and division of natural numbers.

1.2.2 Whole Numbers

(i) When a natural number is subtracted from itself we can not say what is the left out number. To remove this difficulty, the natural numbers were extended by the number zero (0), to get what is called the system of whole numbers

Thus, the whole numbers are

0, 1, 2, 3, ...

Again, like before, there is no greatest whole number.

(ii) The number 0 has the following properties:

$a + 0 = a = 0 + a$

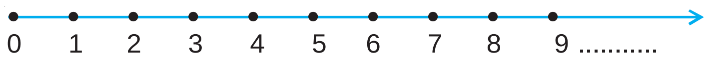
$a - 0 = a$ but $(0 - a)$ is not defined in whole numbers

$a \times 0 = 0 = 0 \times a$

Division by zero (0) is not defined.

(iii) Four fundamental operations can be performed on whole numbers also as in the case of natural numbers (with restrictions for subtraction and division).

(iv) Whole numbers can also be represented on the number line as follows:



1.2.3 Integers

While dealing with natural numbers and whole numbers we found that it is not always possible to subtract a number from another.

For example, (2 – 3), (3 – 7), (9 – 20) etc. are all not possible in the system of natural numbers and whole numbers. Thus, it needed another extension of numbers which allow such subtractions.

Thus, we extend whole numbers by such numbers as –1 (called negative 1), – 2 (negative 2) and so on such that

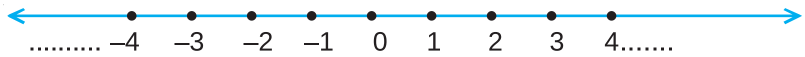
$$1 + (-1) = 0,\ 2 + (-2) = 0,\ 3 + (-3) = 0...,\ 99 + (-99) = 0,\ ...$$

Thus, we have extended the whole numbers to another system of numbers, called integers. The integers therefore are

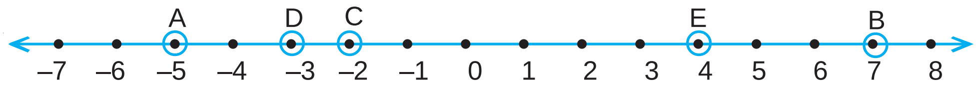
$$...,\ -7,\ -6,\ -5,\ -4,\ -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ ...$$

1.2.4 Representing Integers on the Number Line

We extend the number line used for representing whole numbers to the left of zero and mark points – 1, – 2, – 3, – 4, ... such that 1 and – 1, 2 and – 2, 3 and – 3 are equidistant from zero and are in opposite directions of zero. Thus, we have the integer number line as follows:



We can now easily represent integers on the number line. For example, let us represent – 5, 7, – 2, – 3, 4 on the number line. In the figure, the points A, B, C, D and E respectively represent – 5, 7, – 2, – 3 and 4.



We note here that if an integer a > b, then 'a' will always be to the right of 'b', otherwise vise-versa.

For example, in the above figure 7 > 4, therefore B lies to the right of E. Similarly, – 2 > – 5, therefore C (– 2) lies to the right of A (–5).

Conversely, as 4 < 7, therefore 4 lies to the left of 7 which is shown in the figure as E is to the left of B

∴ For finding the greater (or smaller) of the two integers a and b, we follow the following rule:

i) a > b, if a is to the right of b

ii) a < b, if a is to the left of b

Example 1.1: Identify natural numbers, whole numbers and integers from the following:-

15, 22, – 6, 7, – 13, 0, 12, – 12, 13, – 31

Solution: Natural numbers are: 7, 12, 13, 15 and 22

whole numbers are: 0, 7, 12, 13, 15 and 22

Integers are: – 31, – 13, – 12, – 6, 0, 7, 12, 13, 15 and 22

Example 1.2: From the following, identify those which are (i) not natural numbers (ii) not whole numbers

– 17, 15, 23, – 6, – 4, 0, 16, 18, 22, 31

Solution: i) – 17, – 6, – 4 and 0 are not natural numbers

ii) – 17, – 6, – 4 are not whole numbers

Note: From the above examples, we can say that

i) all natural numbers are whole numbers and integers also but the vice-versa is not true

ii) all whole numbers are integers also

You have studied four fundamental operations on integers in earlier classes. Without repeating them here, we will take some examples and illustrate them here

Example 1.3: Simplify the following and state whether the result is an integer or not

$12 \times 4,\ 7 \div 3,\ 18 \div 3,\ 36 \div 7,\ 14 \times 2,\ 18 \div 36,\ 13 \times (-3)$

Solution: $12 \times 4 = 48$; it is an integer

$7 \div 3 = \frac{7}{3}$; It is not an integer

$18 \div 3 = 6$; It is an integer

$36 \div 7 = \frac{36}{7}$; It is not an integer.

$14 \times 2 = 28$, It is an integer

$18 \div 36 = \frac{18}{36}$; It is not an integer

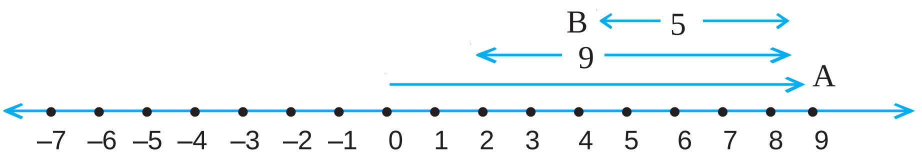
$13 \times (-3) = -39$; It is an integer

Example 1.4: Using number line, add the following integers:

(i) $9 + (-5)$ (ii) $(-3) + (-7)$

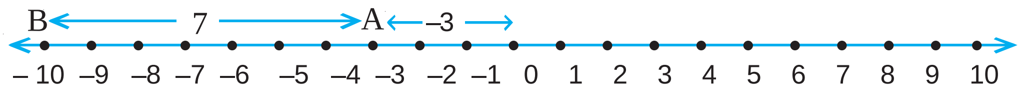
Solution:

(i)



A represents 9 on the number line. Going 5 units to the left of A, we reach the point B, which represents 4.

$\therefore\ 9 + (-5) = 4$

(ii) 

Starting from zero (0) and going three units to the left of zero, we reach the point A, which represents – 3. From A going 7 units to the left of A, we reach the point B which represents – 10.

$\therefore$ (–3) + (–7) = – 10

1.3 RATIONAL NUMBERS

Consider the situation, when an integer a is divided by another non-zero integer b. The following cases arise:

(i) When ‘a’ is a multiple of ‘b’

Suppose a = mb, where m is a natural number or integer, then $\frac{a}{b} = \text{m}$

(ii) When a is not a multiple of b

In this case $\frac{a}{b}$ is not an integer, and hence is a new type of number. Such a number is called a rational number.

Thus, a number which can be put in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$, is called a rational number

Thus, $-\frac{2}{3}, \frac{5}{-8}, \frac{6}{2}, \frac{11}{7}$ are all rational numbers.

1.3.1 Positive and Negative Rational Numbers

(i) A rational number $\frac{p}{q}$ is said to be a positive rational number if p and q are both positive or both negative integers

Thus $\frac{3}{4}, \frac{5}{6}, \frac{-3}{-2}, \frac{-8}{-6}, \frac{-12}{-57}$ are all positive rationals.

(ii) If the integes p and q are of different signs, then $\frac{p}{q}$ is said to be a negaive rational number.

Thus, $\frac{-7}{2}, \frac{6}{-5}, \frac{-12}{4}, \frac{16}{-3}$ are all negaive rationals.

1.3.2 Standard form of a Rational Number

We know that numbers of the form

$$\frac{-p}{q}, \frac{p}{-q}, \frac{-p}{-q} \text{ and } \frac{p}{q}$$

are all rational numbers, where p and q are positive integers

We can see that

$$\frac{-p}{q} = -\left(\frac{p}{q}\right), \frac{-p}{-q} = \frac{-(-p)}{-(-q)} = \frac{p}{q}, \frac{p}{-q} = \frac{(-p)}{-(-q)} = \frac{-p}{q},$$

In each of the above cases, we have made the denominator q as positive.

A rational number $\frac{p}{q}$, where p and q are integers and $q \neq 0$, in which q is positive (or made positive) and p and q are co-prime (i.e. when they do not have a common factor other than 1 and –1) is said to be in standard form.

Thus the standard form of the rational number $\frac{2}{-3}$ is $\frac{-2}{3}$. Similarly, $\frac{-5}{6}$ and $\frac{-3}{5}$ are rational numbers in standard form.

Note: "A rational number in standard form is also referred to as "a rational number in its lowest form". In this lesson, we will be using these two terms interchangably.

For example, rational number $\frac{18}{27}$ can be written as $\frac{2}{3}$ in the standard form (or the lowest form) .

Similarly, $\frac{25}{-35}$, in standard form (or in lowest form) can be written as $\frac{-5}{7}$ (cancelling out 5 from both numerator and denominator).

Some Important Results

(i) Every natural number is a rational number but the vice-versa is not always true.

(ii) Every whole number and integer is a rational number but vice-versa is not always true.

Example 1.5: Which of the following are rational numbers and which are not?

$$-2, \frac{5}{3}, -17, \frac{15}{7}, \frac{18}{5}, -\frac{7}{6}$$

Solution:

(i) –2 can be written as $\frac{-2}{1}$, which is of the form $\frac{p}{q}$, $q \neq 0$. Therefore, –2 is a rational number.

(ii) $\frac{5}{3}$ is a rational number, as it is of the form $\frac{p}{q}$, $q \neq 0$

(iii) –17 is also a rational number as it is of the form $\frac{-17}{1}$

(iv) Similarly, $\frac{15}{7}, \frac{18}{5}$ and $\frac{-7}{6}$ are all rational numbers according to the same argument

Example 1.6: Write the following rational numbers in their lowest terms:

(i) $\frac{-24}{192}$ (ii) $\frac{12}{168}$ (iii) $\frac{-21}{49}$

Solution:

(i) $\frac{-24}{192} = \frac{-3 \times 8}{3 \times 8 \times 8} = \frac{-1}{8}$

$-\frac{1}{8}$ is the lowest form of the rational number $\frac{-24}{192}$

(ii) $\frac{12}{168} = \frac{12}{12 \times 14} = \frac{1}{14}$

$\therefore \frac{1}{14}$ is the lowest form of the rational number $\frac{12}{168}$

(iii) $\frac{-21}{49} = \frac{-3 \times 7}{7 \times 7} = \frac{-3}{7}$

$\therefore \frac{-3}{7}$ is the lowest form of the rational number $\frac{-21}{49}$

1.4 EQUIVALENT FORMS OF A RATIONAL NUMBER

A rational number can be written in an equivalent form by multiplying/dividing the numerator and denominator of the given rational number by the same number.

For example

$$\frac{2}{3}=\frac{2\times2}{3\times2}=\frac{4}{6},\quad \frac{2}{3}=\frac{2\times4}{3\times4}=\frac{8}{12},\qquad \frac{2\times8}{3\times8}=\frac{16}{24}$$

$\therefore \frac{4}{6}$, $\frac{8}{12}$, $\frac{16}{24}$ etc. are equivalent forms of the rational number $\frac{2}{3}$

Similarly

$$\frac{3}{8}=\frac{6}{16}=\frac{21}{56}=\frac{27}{72}=\ldots$$

and $$\frac{4}{7}=\frac{8}{14}=\frac{12}{21}=\frac{28}{49}=\ldots$$

are equivalent forms of $\frac{3}{8}$ and $\frac{4}{7}$ respectively.

Example 1.7: Write five equivalent forms of the following rational numbers:

(i) $\frac{3}{17}$ (ii) $\frac{-5}{9}$

Solution:

(i) $$\frac{3}{17}=\frac{3\times2}{17\times2}=\frac{6}{34},\ \frac{3}{17}=\frac{3\times4}{17\times4}=\frac{12}{68},\ \frac{3\times(-3)}{17\times(-3)}=\frac{-9}{-51}$$

$$\frac{3\times8}{17\times8}=\frac{24}{136},\ \frac{3}{17}\times\frac{7}{7}=\frac{21}{119}$$

$\therefore$ Five equivalent forms of $\frac{3}{17}$ are

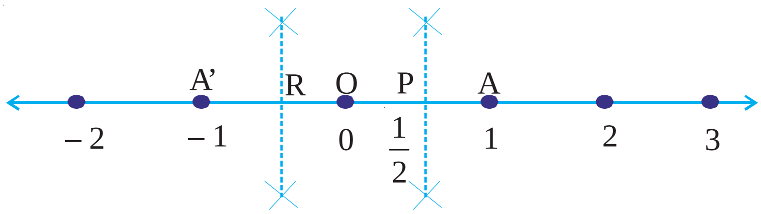
$$\frac{6}{34},\frac{12}{68},\frac{-9}{-51},\frac{24}{136},\frac{21}{119}$$

(ii) As in part (i), five equivalent forms of $\frac{-5}{9}$ are

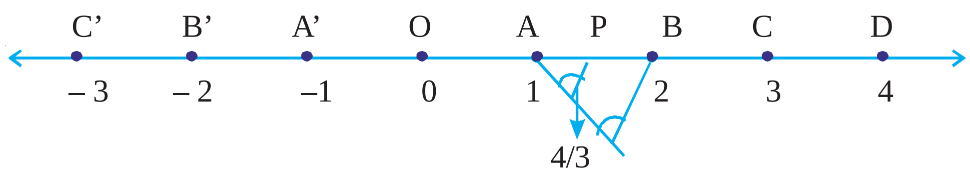
$$\frac{-10}{18}, \frac{-15}{27}, \frac{-20}{36}, \frac{-60}{108}, \frac{-35}{63}$$

1.5 RATIONAL NUMBERS ON THE NUMBER LINE

We know how to represent integers on the number line. Let us try to represent $\frac{1}{2}$ on the number line. The rational number $\frac{1}{2}$ is positive and will be represented to the right of zero. As $0 < \frac{1}{2} < 1$, $\frac{1}{2}$ lies between 0 and 1. Divide the distance OA in two equal parts. This can be done by bisecting OA at P. Let P represent $\frac{1}{2}$. Similarly R, the mid-point of OA', represents the rational number $-\frac{1}{2}$.



Similarly, $\frac{4}{3}$ can be represented on the number line as below:



As $1 < \frac{4}{3} < 2$, therefore $\frac{4}{3}$ lies between 1 and 2. Divide the distance AB in three equal parts. Let one of this part be AP

Now $\frac{4}{3} = 1 + \frac{1}{3} = OA + AP = OP$

The point P represents $\frac{4}{3}$ on the number line.

1.6 COMPARISON OF RATIONAL NUMBERS

In order to compare two rational numbers, we follow any of the following methods:

(i) If two rational numbers, to be compared, have the same denominator, compare their numerators. The number having the greater numerator is the greater rational number.

Thus for the two rational numbers $\frac{5}{17}$ and $\frac{9}{17}$, with the same positive denominator

17, $\frac{9}{17} > \frac{5}{17}$ as $9 > 5$

$\therefore \frac{9}{17} > \frac{5}{17}$

(ii) If two rational numbers are having different denominators, make their denominators equal by taking their equivalent form and then compare the numerators of the resulting rational numbers. The number having a greater numerator is greater rational number.

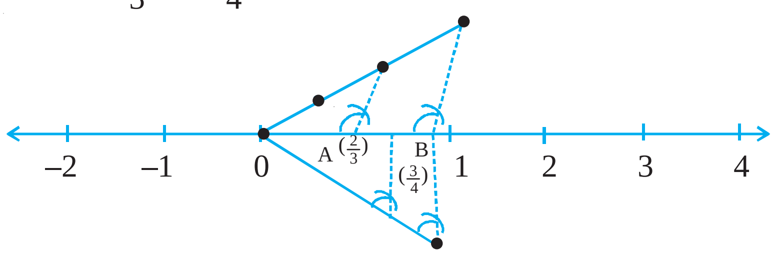
For example, to compare two rational numbers $\frac{3}{7}$ and $\frac{6}{11}$, we first make their denominators same in the following manner:

$\frac{3 \times 11}{7 \times 11} = \frac{33}{77}$ and $\frac{9 \times 7}{11 \times 7} = \frac{42}{77}$

As $42 > 33$, $\frac{42}{77} > \frac{33}{77}$ or $\frac{6}{11} > \frac{3}{7}$

(iii) By plotting two given rational numbers on the number line we see that the rational number to the right of the other rational number is greater.

For example, take $\frac{2}{3}$ and $\frac{3}{4}$, we plot these numbers on the number line as below:



$0 < \frac{2}{3} < 1$ and $0 < \frac{3}{4} < 1$. It means $\frac{2}{3}$ and $\frac{3}{4}$ both lie between 0 and 1. By the method of dividing a line into equal number of parts, A represents $\frac{2}{3}$ and B represents $\frac{3}{4}$

As B is to the right of A, $\frac{3}{4} > \frac{2}{3}$ or $\frac{2}{3} < \frac{3}{4}$

$\therefore$ Out of $\frac{2}{3}$ and $\frac{3}{4}$, $\frac{3}{4}$ is the greater number.

CHECK YOUR PROGRESS 1.1

1. Identify rational numbers and integers from the following:

   $4, \frac{-3}{4}, \frac{5}{6}, -36, \frac{12}{7}, \frac{3}{-8}, \frac{15}{7}, -6$

2. From the following identify those which are not :

   (i) natural numbers

   (ii) whole numbers

   (iii) integers

   (iv) rational numbers

   $-\frac{7}{4}, 16, \frac{-3}{7}, -15, 0, \frac{5}{17}, \frac{3}{-4}, -\frac{4}{3}$

3. By making the following rational numbers with same denominator, simplify the following and specify whether the result in each case is a natural number, whole number, integer or a rational number:

   (i) $3 + \frac{7}{3}$ (ii) $-3 + \frac{10}{4}$ (iii) $-8 - 13$ (iv) $12 - 12$

   (v) $\frac{9}{2} - \frac{1}{2}$ (vi) $2 \times \frac{5}{7}$ (vii) $8 \div 3$

4. Use the number line to add the following:-

   (i) $9 + (-7)$ (ii) $(-5) + (-3)$ (iii) $(-3) + (4)$

5. Which of the following are rational numbers in lowest term?

$$\frac{8}{12}, \frac{5}{7}, \frac{-3}{12}, \frac{-6}{7}, \frac{2\sqrt{3}}{\sqrt{27}}, \frac{15}{24}$$

6. Which of the following rational numbers are integers?

$$-10, \frac{15}{5}, \frac{-5}{15}, \frac{13}{5}, \frac{27}{9}, \frac{7\times 3}{14}, \frac{-6}{-2}$$

7. Write 3 rational numbers equivalent to given rational numbers:

$$\frac{2}{5}, \frac{-5}{6}, \frac{17}{3}$$

8. Represent the following rational numbers on the number line.

$$\frac{2}{5}, \frac{3}{4}, \frac{1}{2}$$

9. Compare the following rational numbers by (i) changing them to rational numbers in equivalent forms (ii) using number line:

(a) $\frac{2}{3}$ and $\frac{3}{4}$ (b) $\frac{3}{5}$ and $\frac{7}{9}$ (c) $\frac{-2}{3}$ and $\frac{-1}{2}$

(d) $\frac{3}{7}$ and $\frac{5}{11}$ (e) $\frac{-7}{6}$ and $\frac{3}{2}$

1.7 FOUR FUNDAMENTAL OPERATIONS ON RATIONAL NUMBERS

1.7.1 Addition of Rational Numbers

(a) Consider the addition of rational numbers $\frac{p}{q}, \frac{r}{q}$

$$\frac{p}{q}+\frac{r}{q}=\frac{p+r}{q}$$

For example

$$\text{(i)}\quad \frac{2}{3}+\frac{5}{3}=\frac{2+5}{3}=\frac{7}{3}$$

$$\text{(ii)}\quad \frac{3}{17}+\frac{9}{17}=\frac{3+9}{17}=\frac{12}{17}$$

and

$$\text{(iii)}\quad \frac{14}{3}+\left(\frac{-5}{3}\right)=\frac{14-5}{3}=\frac{9}{3}=3$$

(b) Consider the two rational numbers $\frac{p}{q}$ and $\frac{r}{s}$.

$$\frac{p}{q}+\frac{r}{s}=\frac{ps}{qs}+\frac{rq}{sq}=\frac{ps+rq}{qs}$$

For example,

$$\text{(i)}\ \frac{3}{4}+\frac{2}{3}=\frac{3\times3+4\times2}{4\times3}=\frac{9+8}{12}=\frac{17}{12}$$

$$\text{(ii)}\ -\frac{4}{5}+\frac{7}{8}=\frac{-4\times8+5\times7}{5\times8}=\frac{35-32}{40}=\frac{3}{40}$$

From the above two cases, we generalise the following rule:

(a) The addition of two rational numbers with common denominator is the rational number with common denominator and numerator as the sum of the numerators of the two rational numbers.

(b) The sum of two rational numbers with different denominators is a rational number with the denominator equal to the product of the denominators of two rational numbers and the numerator equal to sum of the product of the numerator of first rational number with the denominator of second and the product of numerator of second rational number and the denominator of the first rational number.

Let us take sone examples:

Example 1.8: Add the following rational numbers:

(i) $\frac{2}{7}$ and $\frac{6}{7}$ (ii) $\frac{4}{17}$ and $\frac{-3}{17}$ (iii) $-\frac{5}{11}$ and $\frac{-3}{11}$

Solution: (i) $\frac{2}{7}+\frac{6}{7}=\frac{2+6}{7}=\frac{8}{7}$

$$\therefore \frac{2}{7}+\frac{6}{7}=\frac{8}{7}$$

(ii) $\frac{4}{17}+\frac{(-3)}{17}=\frac{4+(-3)}{17}=\frac{4-3}{17}=\frac{1}{17}$

$$\therefore \frac{4}{17}+\frac{(-3)}{17}=\frac{1}{17}$$

(iii) $\left(-\frac{5}{11}\right)+\left(\frac{-3}{11}\right)=\frac{(-5)+(-3)}{11}=\frac{-5-3}{11}=\frac{-8}{11}$

$$\therefore \left(-\frac{5}{11}\right)+\left(-\frac{3}{11}\right)=-\frac{8}{11}$$

Example 1.9: Add each of the following rational numbers:

(i) $\frac{3}{4}$ and $\frac{1}{7}$ (ii) $\frac{2}{7}$ and $\frac{3}{5}$ (iii) $\frac{5}{9}$ and $-\frac{4}{15}$

Solution: (i) We have $\frac{3}{4}+\frac{1}{7}$

$$=\frac{3\times 7}{4\times 7}+\frac{1\times 4}{7\times 4}$$

$$=\frac{21}{28}+\frac{4}{28}=\frac{21+4}{28}$$

$$=\frac{25}{28}$$

$$\therefore \frac{3}{4}+\frac{1}{7}=\frac{25}{28} \text{ or } \left[\frac{3\times 7+4\times 1}{4\times 7}=\frac{21+4}{28}=\frac{25}{28}\right]$$

(ii) $\frac{2}{7}+\frac{3}{5}$

$$=\frac{2\times 5}{7\times 5}+\frac{3\times 7}{5\times 7}$$

$$=\frac{10}{35}+\frac{21}{35}$$

$$=\frac{10+21}{35}=\frac{31}{35}$$

$$\therefore \frac{2}{7}+\frac{3}{5}=\frac{31}{35} \text{ or } \left[\frac{2\times 5+3\times 7}{35}=\frac{10+21}{35}=\frac{31}{35}\right]$$

(iii) $\frac{5}{9}+\frac{(-4)}{15}$

$$=\frac{5\times 15}{9\times 15}+\frac{(-4)\times 9}{15\times 9}$$

$$=\frac{75}{135}+\frac{(-36)}{135}$$

$$= \frac{75-36}{135} = \frac{39}{135} = \frac{3\times13}{3\times45} = \frac{13}{45}$$

$$\therefore \frac{5}{9} + \frac{(-4)}{15} = \frac{13}{45} \text{ or } \left[\frac{5\times15+9\times(-4)}{9\times15} = \frac{75-36}{135} = \frac{39}{135} = \frac{13}{45}\right]$$

1.7.2 Subtraction of Rational Numbers

(a) $\frac{p}{q} - \frac{r}{q} = \frac{p-r}{q}$

(b) $\frac{p}{q} - \frac{r}{s} = \frac{ps-qr}{qs}$

Example 1.10: Simplify the following:

(i) $\frac{7}{4} - \frac{1}{4}$ (ii) $\frac{3}{5} - \frac{2}{12}$

Solution: (i) $\frac{7}{4} - \frac{1}{4} = \frac{7-1}{4} = \frac{6}{4} = \frac{2\times3}{2\times2} = \frac{3}{2}$

(ii) $\frac{3}{5} - \frac{2}{12} = \frac{3\times12}{5\times12} - \frac{2\times5}{12\times5}$

$$= \frac{36}{60} - \frac{10}{60} = \frac{36-10}{60}$$

$$= \frac{26}{60} = \frac{13\times2}{30\times2} = \frac{13}{30}$$

1.7.3 Multiplication and Division of Rational Numbers

(i) Multiplication of two rational number $\left(\frac{p}{q}\right)$ and $\left(\frac{r}{s}\right)$, $q \neq 0$, $s \neq 0$ is the rational number $\frac{pr}{ps}$ where $qs \neq 0$

$$= \frac{\text{product of numerators}}{\text{product of denominators}}$$

(ii) Division of two rational numbers $\frac{p}{q}$ and $\frac{r}{s}$, such that $q \neq 0$, $s \neq 0$, is the rational number $\frac{ps}{qr}$, where $qr \neq 0$

In other words $\left(\frac{p}{q}\right) \div \left(\frac{r}{s}\right) = \frac{p}{q} \times \left(\frac{s}{r}\right)$

Or (First rational number) × (Reciprocal of the second rational number)

Let us consider some examples.

Example 1.11: Multiply the following rational numbers:

(i) $\frac{3}{7}$ and $\frac{2}{9}$ (ii) $\frac{5}{6}$ and $\left(\frac{-2}{19}\right)$ (iii) $\frac{7}{13}$ and $\left(\frac{-2}{-5}\right)$

Solution: (i) $\frac{3}{7} \times \frac{2}{9} = \frac{3 \times 2}{7 \times 9} = \frac{3 \times 2}{7 \times 3 \times 3} = \frac{2}{21}$

$$\therefore \left(\frac{3}{7}\right) \times \left(\frac{2}{9}\right) = \frac{2}{21}$$

(ii) $\frac{5}{6} \times \left(\frac{-2}{19}\right) = \frac{5 \times (-2)}{6 \times 19}$

$$= -\frac{2 \times 5}{2 \times 3 \times 19} = -\frac{5}{57}$$

$$\therefore \left(\frac{5}{6}\right) \times \left(-\frac{2}{19}\right) = -\frac{5}{57}$$

(iii) $\frac{7}{13} \times \left(\frac{-2}{-5}\right) = \left(\frac{7}{13}\right)\left(\frac{-(-2)}{5}\right)$

$$= \frac{7}{13} \times \frac{2}{5} = \frac{7 \times 2}{13 \times 5} = \frac{14}{65}$$

$$\therefore \left(\frac{7}{13}\right) \times \left(\frac{-2}{-5}\right) = \frac{14}{65}$$

Example 1.12: Simply the following:

(i) $\left(\frac{3}{4}\right) \div \left(\frac{7}{12}\right)$ (ii) $\frac{9}{16} \div \left(-\frac{105}{12}\right)$ (iii) $\left(\frac{87}{27}\right) \div \left(\frac{29}{18}\right)$

Solution: (i) $\left(\frac{3}{4}\right)\div\left(\frac{7}{12}\right)$

$$=\left(\frac{3}{4}\right)\times\left(\frac{12}{7}\right) \qquad \left[\text{Reciprocal of } \frac{7}{12} \text{ is } \frac{12}{7}\right]$$

$$=\frac{3\times12}{4\times7}=\frac{3\times3\times4}{7\times4}=\frac{9}{7}$$

$$\therefore \left(\frac{3}{4}\right)\div\left(\frac{7}{12}\right)=\frac{9}{7}$$

(ii) $\left(\frac{9}{16}\right)\div\left(\frac{-105}{2}\right)$

$$\left(\frac{9}{16}\right)\times\left(\frac{2}{-105}\right) \qquad \left[\text{Reciprocal of } \frac{-105}{2} \text{ is } \frac{2}{-105}\right]$$

$$=-\frac{9\times2}{2\times8\times3\times35}=-\frac{3\times3\times2}{2\times8\times3\times35}$$

$$=\frac{-3}{8\times35}=\frac{-3}{280}$$

$$\therefore \left(\frac{9}{16}\right)\div\left(\frac{-105}{2}\right)=\frac{-3}{280}$$

(iii) $\left(\frac{87}{27}\right)\div\left(\frac{29}{18}\right)$

$$=\left(\frac{87}{27}\right)\times\left(\frac{18}{29}\right)=\frac{87}{27}\times\frac{18}{29}=\frac{29\times3\times2\times9}{9\times3\times29}=\frac{2}{1}$$

$$\therefore \left(\frac{87}{27}\right)\div\left(\frac{29}{18}\right)=\frac{2}{1}$$

CHECK YOUR PROGRESS 1.2

1. Add the following rational numbers:

   (i) $\frac{3}{7}, \frac{6}{7}$ (ii) $\frac{2}{15}, -\frac{6}{15}$ (iii) $\frac{3}{20}, \frac{-7}{-20}$ (iv) $\frac{1}{8}, \frac{3}{8}$

2. Add the following rational numbers:

   (i) $\frac{3}{2}, \frac{5}{3}$ (ii) $\frac{17}{7}, \frac{5}{9}$ (iii) $\frac{2}{5}, \frac{-5}{7}$

3. Perform the indicated operations:

   (i) $\left(-\frac{7}{8}+\frac{-5}{12}\right)+\frac{3}{16}$ (ii) $\left(\frac{7}{3}+\frac{3}{4}\right)+\left(-\frac{3}{5}\right)$

4. Subtract:-

   (i) $\frac{7}{15}$ from $\frac{13}{15}$ (ii) $\frac{7}{3}$ from $-\frac{5}{3}$ (iii) $\frac{3}{7}$ from $\frac{9}{24}$

5. Simplify:-

   (i) $\left(3\frac{1}{5}+\frac{7}{5}-2\frac{1}{6}\right)$ (ii) $\frac{5}{2}+\frac{13}{4}-6\frac{3}{4}$

6. Multiply:-

   (i) $\frac{2}{11}$ by $\frac{5}{6}$ (ii) $-\frac{3}{11}$ by $\frac{-33}{35}$ (iii) $\frac{-11}{3}$ by $\frac{-27}{77}$

7. Divide:

   (i) $\frac{1}{2}$ by $\frac{1}{4}$ (ii) $\frac{-7}{4}$ by $\frac{-4}{5}$ (iii) $\frac{35}{33}$ by $\frac{-7}{22}$

8. Simplify the following:

   (i) $\left(\frac{2}{3}+\frac{7}{8}\right)\times\frac{8}{25}\div\frac{37}{15}$ (ii) $\left[\left(\frac{3}{4}-\frac{2}{3}\right)\div\frac{1}{4}\right]\times 21$

9. Divide the sum of $\frac{16}{7}$ and $\frac{-3}{14}$ by their difference.

10. A number when multiplied by $\frac{13}{3}$ gives $\frac{39}{12}$. Find the number.

1.8 DECIMAL REPRESENTATION OF A RATIONAL NUMBER

You are familiar with the division of an integer by another integer and expressing the result as a decimal number. The process of expressing a rational number into decimal form is to carryout the process of long division using decimal notation.

Let us consider some examples.

Example 1.13: Represent each one of the following into a decimal number:

(i) $\frac{12}{5}$ (ii) $\frac{-27}{25}$ (iii) $\frac{13}{16}$

Solution: i) Using long division, we get

$$\begin{array}{r|l} & 2.4 \\ 5 & 12.0 \\ & \underline{10} \\ & 2.0 \\ & \underline{2.0} \\ & \times \end{array}$$

Hence, $\frac{12}{5} = 2.4$

ii)

$$\begin{array}{r|l} 25 & -27\ (-1.08 \\ & \underline{25} \\ & 200 \\ & \underline{200} \\ & \times \end{array}$$

Hence, $\frac{-27}{25} = -1.08$

iii)

$$\begin{array}{r|l} & 0.8125 \\ 16 & 13.0000 \\ & \underline{128} \\ & 20 \\ & \underline{16} \\ & 40 \\ & \underline{32} \\ & 80 \\ & \underline{80} \\ & \times \end{array}$$

Hence, $\frac{13}{16} = 0.8125$

From the above examples, it can be seen that the division process stops after a finite number of steps, when the remainder becomes zero and the resulting decimal number has a finite number of decimal places. Such decimals are known as **terminating decimals**.

Note: Note that in the above division, the denominators of the rational numbers had only 2 or 5 or both as the only prime factors.

Alternatively, we could have written $\frac{12}{5}$ as $\frac{12\times 2}{5\times 2} = \frac{24}{10} = 2.4$ and similarly for the others

Let us consider another example.

Example 14: Write the decimal representation of each of the following:

(a) $\frac{7}{3}$ (b) $\frac{2}{7}$ (c) $\frac{5}{11}$

Solution: (a)

$$\begin{array}{r} 2.33 \\ 3\overline{)7.00} \\ \underline{6} \\ 1.0 \\ \underline{9} \\ 1.0 \\ \underline{9} \\ 1.00 \end{array}$$

Here the remainder 1 repeats.

$\therefore$ The decimal is not a terminating decimal

$\frac{7}{3} = 2.333...$ or $2.\overline{3}$

(b)

$$\begin{array}{r} 0.28571428 \\ 7\overline{)2.000} \\ \underline{14} \\ 60 \\ \underline{56} \\ 40 \\ \underline{35} \\ 50 \\ \underline{49} \\ 10 \\ \underline{7} \\ 30 \\ \underline{28} \\ 20 \\ \underline{14} \\ 60 \\ \underline{56} \\ 4 \end{array}$$

Here when the remainder is 3, the digit after that start repeating

$\frac{2}{7} = 0.\overline{285714}$

Note: A bar over a digit or a group of digits implies that digit or that group of digits starts repeating itself indefinitely.

(c)

$$\begin{array}{r} 0.454 \\ 11\overline{)5.00} \\ \underline{44} \\ 60 \\ \underline{55} \\ 50 \\ \underline{44} \\ 50... \end{array}$$

Here again when the remainder is 5, the digits after 5 start repeating

$\therefore \frac{5}{11} = 0.\overline{45}$

From the above, it is clear that in cases where the denominator has factors other than 2 or 5, the decimal representation starts repeating. Such decimals are called non-terminating repeating decimals.

Thus, we see from examples 1.13 and 1.14 that the decimal representation of a rational number is

(i) either a terminating decimal (and the remainder is zero after a finite number of steps)

(ii) or a non-terminating repeating decimal (where the division will never end)

$\therefore$ Thus, a rational number is either a terminating decimal or a non-terminating repeating decimal

1.8 EXPRESSING DECIMAL EXPANSION OF A RATIONAL NUMBER IN p/q FORM

Let us explain it through examples

Example 1.15: Express (i) 0.48 and (ii) 0.1357 in $\frac{p}{q}$ form

Solution: (i) $0.48 = \frac{48}{100} = \frac{12}{25}$

(ii) $0.1375 = \frac{1375}{10000} = \frac{55}{400} = \frac{11}{80}$

Example 1.16: Express (i) 0.666... (ii) 0.374374... in $\frac{p}{q}$ form

Solution: (i) Let $x = 0.666...$ (A)

$\therefore\ 10\,x = 6.666...$ (B)

(B) – (A) gives $9\,x = 6$ or $x = \frac{2}{3}$

(ii) Let $x = 0.374374374....$ (A)

$1000\,x = 374.374374374....$ (B)

(B) – (A) gives $999\,x = 374$

or $x = \frac{374}{999}$

$$\therefore 0.374374374... = \frac{374}{999}$$

The above example illustrates that:

A terminating decimal or a non-terminating recurring decimal represents a rational number

Note: The non- terminating recurring decimals like 0.374374374... are written as $0.\overline{374}$. The bar on the group of digits 374 indicate that the group of digits repeats again and again.

CHECK YOUR PROGRESS 1.3

1. Represent the following rational numbers in the decimal form:

   (i) $\frac{31}{80}$ (ii) $\frac{12}{25}$ (iii) $\frac{12}{8}$ (iv) $\frac{75}{12}$ (v) $\frac{91}{63}$

2. Represent the following rational numbers in the decimal form:

   (i) $\frac{2}{3}$ (ii) $\frac{5}{7}$ (iii) $\frac{25}{11}$

3. Represent the following decimals in the form $\frac{p}{q}$.

   (a) (i) 2.3 (ii) – 3.12 (iii) –0.715 (iv) 8.146

   (b) (i) $0.\overline{333}$ (ii) $3.\overline{42}$ (iii) – 0.315315315...

1.9 RATIONAL NUMBERS BETWEEN TWO RATIONAL NUMBERS

Is it possible to find a rational number between two given rational numbers. To explore this, consider the following examples.

Example 1.17: Find a rational number between $\frac{3}{4}$ and $\frac{6}{5}$

Solution: Let us try to find the number $\frac{1}{2}\left(\frac{3}{4}+\frac{6}{5}\right)$

$$= \frac{1}{2}\left(\frac{15+24}{20}\right) = \frac{39}{40}$$

Now $\frac{3}{4} = \frac{3\times10}{4\times10} = \frac{30}{40}$

and $\frac{6}{5} = \frac{6\times8}{5\times8} = \frac{48}{40}$

Obviously $\frac{30}{40} < \frac{39}{40} < \frac{48}{40}$

i.e. $\frac{39}{40}$ is a rational number between the rational numbers $\frac{3}{4}$ and $\frac{6}{5}$.

Note: $\frac{3}{4} = 0.75$, $\frac{39}{40} = 0.975$ and $\frac{6}{5} = 1.2$

$\therefore 0.75 < 0.975 < 1.2$

or $\frac{3}{4} < \frac{39}{40} < \frac{6}{5}$

$\therefore$ This can be done by either way:

(i) reducing each of the given rational number with a common base and then taking their average

or (ii) by finding the decimal expansions of the two given rational numbers and then taking their average.

The question now arises, "How many rationals can be found between two given rationals? Consider the following examples.

Example 1.18: Find 3 rational numbers between $\frac{1}{2}$ and $\frac{3}{4}$.

Solution: $\frac{1}{2} = \frac{1\times8}{2\times8} = \frac{8}{16}$

and $\frac{3}{4} = \frac{3\times4}{4\times4} = \frac{12}{16}$

As $\frac{8}{16} < \frac{9}{16} < \frac{10}{16} < \frac{11}{16} < \frac{12}{16}$

$\therefore$ We have been able to find 3 rational numbers

$$\frac{9}{16}, \frac{10}{16} \text{ and } \frac{11}{16} \text{ between } \frac{1}{2} \text{ and } \frac{3}{4}$$

In fact, we can find any number of rationals between two given numbers.

Again $\frac{1}{2} = \frac{50}{2\times 50} = \frac{50}{100}$

$$\frac{3}{4} = \frac{3\times 25}{4\times 25} = \frac{75}{100}$$

As $\frac{50}{100} < \frac{51}{100} < \frac{52}{100} < \frac{53}{100} < \ldots\ldots < \frac{72}{100} < \frac{73}{100} < \frac{74}{100} < \frac{75}{100} < \ldots.$ (i)

$\therefore$ we have been able to find 24 rational numbers between $\frac{1}{2}$ and $\frac{3}{4}$ as given in (i) above.

We can continue in this way further.

Note: From the above it is clear that between any two rationals an infinite number of rationals can be found.

CHECK YOUR PROGRESS 1.4

1. Find a rational number between the following rational numbers:

   (i) $\frac{3}{4}$ and $\frac{4}{3}$ (ii) 5 and 6 (iii) $-\frac{3}{4}$ and $\frac{1}{3}$

2. Find two rational numbers between the following rational numbers:

   (i) $-\frac{2}{3}$ and $\frac{1}{2}$ (ii) $-\frac{2}{3}$ and $-\frac{1}{4}$

3. Find 5 rational numbers between the following rational numbers:

   (i) 0.27 and 0.30 (ii) 7.31 and 7.35

   (iii) 20.75 and 26.80 (iv) 1.001 and 1.002

1.10 IRRATIONAL NUMBERS

We have seen that the decimal expansion of a rational number is either terminating or is a non-terminating and repeating deimal.

Are there decimals which are neither terminating nor non-terminating but repeating decimals? Consider the following decimal:

0.10 100 1000 10000 1....... (i)

You can see that this decimal has a definite pattern and it can be written indefinitely, and there is no block of digits which is repeating. Thus, it is an example of a non-terminating and non-repeating decimal. A similar decimal is given as under:

0.1 2 3 4 5 6 7 8 9 10 11 12 13..... (ii)

Can you write the next group of digits in (i) and (ii)? The next six digits in (i) are 000001... and in (ii) they are 14 15 16 ...

Such decimals as in (i) and (ii) represent irrational numbers.

Thus, *a decimal expansion which is neither terminating nor is repeating represents an irrational number.*

1.11 INADEQUACY OF RATIONAL NUMBERS

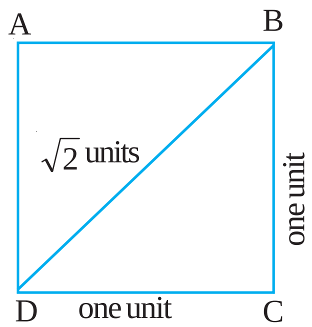
Can we measure all the lengths in terms of rational numbers? Can we measure all weights in terms of rational numbers?

Let us examine the following situation:

Consider a square ABCD, each of whose sides is 1 unit.

Naturally the diagonal BD is of length $\sqrt{2}$ units.

It can be proved that $\sqrt{2}$ is not a rational number, as there is no rational, whose square is 2, [Proof is beyond the scope of this lesson].

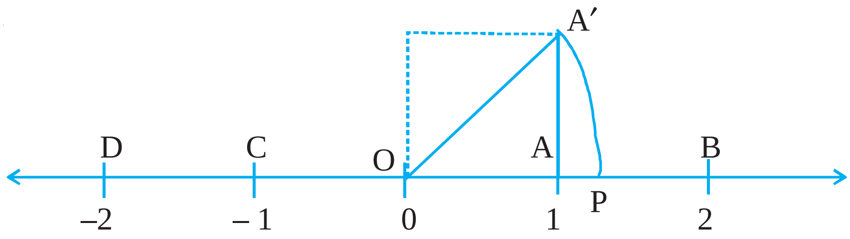


We conclude that we can not exactly measure the lengths of all line-segments using rationals, in terms of a given unit of length. Thus, the rational numbers are inadequate to measure all lengths in terms of a given unit. This inadequacy necessitates the extension of rational numbers to irrationals (which are not rational)

We have also read that corresponding to every rational number, there corresponds a point on the number line. Consider the converse of this statement:

Given a point on the number line, will it always correspond to a rational number? The answer to this question is also "No". For clarifying this, we take the following example.

On the number line take points O, A, B, C and D representing rational 0, 1, 2, –1 and –2 respectively. At A draw AA′ ⊥ to OA such that AA′ = 1 unit



$\therefore OA' = \sqrt{1^2 + 1^2} = \sqrt{2}$ units. Taking O as centre and radius OA′, if we draw an arc, we reach the point P, which represents the number $\sqrt{2}$.

As $\sqrt{2}$ is irrational, we conclude that there are points on the number line (like P) which are not represented by a rational number. Similarly, we can show that we can have points like $\sqrt{3},\ 2\sqrt{3},\ 5\sqrt{2}$ etc, which are not represented by rationals.

$\therefore$ The number line, consisting of points corresponding to rational numbers, has gaps on it. Therefore, the number line consists of points corresponding to rational numbers and irrational numbers both.

We have thus extended the system of rational numbers to include irrational numbers also. The system containing rationals and irrationals both is called the Real Number System.

The system of numbers consisting of all rational and irrational numbers is called the system of real numbers.

CHECK YOUR PROGRESS 1.5

1. Write the first three digits of the decimal representation of the following:

   $\sqrt{2},\ \sqrt{3},\ \sqrt{5}$

2. Represent the following numbers on the real number line:

   (i) $\frac{\sqrt{2}}{2}$ (ii) $1+\sqrt{2}$ (iii) $\frac{\sqrt{3}}{2}$

1.12 FINDING IRRATIONAL NUMBER BETWEEN TWO GIVEN NUMBERS

Let us illustrate the process of finding an irrational number between two given numbers with the help of examples.

Example 1.19: Find an irrational number between 2 and 3.

Solution: Consider the number $\sqrt{2 \times 3}$

We know that $\sqrt{6}$ approximately equals 2.45.

$\therefore$ It lies between 2 and 3 and it is an irrational number.

Example 1.20: Find an irrational number lying between $\sqrt{3}$ and 2.

Solution: Consider the number $\dfrac{\sqrt{3}+2}{2}$

$$= 1 + \frac{\sqrt{3}}{2} \approx 1 + \frac{1.732}{2} = 1.866$$

$$\therefore \quad \frac{\sqrt{3}+2}{2} \approx 1.866 \text{ lies betwen } \sqrt{3} \ (\approx 1.732) \text{ and } 2$$

$\therefore$ The required irrational number is $\dfrac{\sqrt{3}+2}{2}$

CHECK YOUR PROGRESS 1.6

1. Find an irrational number between the following pairs of numbers

   (i) 2 and 4 (ii) $\sqrt{3}$ and 3 (iii) $\sqrt{2}$ and $\sqrt{3}$

2. Can you state the number of irrationals between 1 and 2? Illustrate with three examples.

1.13 ROUNDING OFF NUMBERS TO A GIVEN NUMBER OF DECIMAL PLACES

It is sometimes convenient to write the approximate value of a real number upto a desired number of decimal places. Let us illustrate it by examples.

Example 1.21: Express 2.71832 approximately by rounding it off to two places of decimals.

Solution: We look up at the third place after the decimal point. In this case it 8, which is more than 5. So the approximate value of 2.71832, upto two places of decimal is 2.72.

Example 1.22: Find the approximate value of 12.78962 correct upto 3 places of decimals.

Solution: The fourth place of decimals is 6 (more than 5) so we add 1 to the third place to get the approximate value of 12.78962 correct upto three places of decimals as 12.790.

Thus, we observe that to round off a number to some given number of places, we observe the next digit in the decimal part of the number and proceed as below

(i) If the digit is less than 5, we ignore it and state the answer without it.

(ii) If the digit is 5 or more than 5, we add 1 to the preceeding digit to get the required number upto desired number of decimal places.

CHECK YOUR PROGRESS 1.7

1. Write the approximate value of the following correct upto 3 place of decimals.

   (i) 0.77777 (ii) 7.3259 (iii) 1.0118

   (iv) 3.1428 (v) 1.1413

LET US SUM UP

- Recall of natural numbers, whole numbers, integers with four fundamental operations is done.
- Representation of above on the number line.
- Extension of integers to rational numbers - A rational number is a number which can be put in the form p/q, where p and q are integers and $q \neq 0$.
- When q is made positive and p and q have no other common factor, then a rational number is said to be in standard form or lowest form.
- Two rational numbers are said to be the equivalent form of the number if standard forms of the two are same.
- The rational numbers can be represented on the number line.
- Corresponding to a rational number, there exists a unique point on the number line.
- The rational numbers can be compared by
  - reducing them with the same denominator and comparing their numerators.
  - when represented on the number line, the greater rational number lies to the right of the other.

- As in integers, four fundamental operations can be performed on rational numbers also.
- The decimal representation of a rational number is either terminating or non-terminating repeating.
- There exist infinitely many rational numbers between two rational numbers.
- There are points other than those representing rationals on the number line. That shows inadequacy of system of rational numbers.
- The sytem of rational numbers is extended to real numbers.
- Rationals and irrationals together constitute the system of real numbers.
- We can always find an irrational number between two given numbers.
- The decimal representation of an irrational number is non-terminating non repeating.
- We can find the approximate value of a rational or an irrational number upto a given number of decimals.

TERMINAL EXERCISE

1. From the following pick out:

   (i) natural numbers

   (ii) integers which are not natural numbers

   (iii) rationals which are not natural numbers

   (iv) irrational numbers

   $$-3, 17, \frac{6}{7}, \frac{-3}{8}, 0, -32, \frac{3}{14}, \frac{11}{6}, \sqrt{2}, 2+\sqrt{3}$$

2. Write the following integers as rational numbers:

   (i) – 14 (ii) 13 (iii) 0 (iv) 2

   (v) 1 (vi) –1 (vii) –25

3. Express the following rationals in lowest terms:

   $$\frac{6}{8}, \frac{14}{21}, \frac{-17}{153}, \frac{13}{273}$$

4. Express the following rationals in decimal form:

   (i) $\frac{11}{80}$ (ii) $\frac{8}{25}$ (iii) $\frac{14}{8}$ (iv) $\frac{15}{6}$ (v) $\frac{98}{35}$

(vi) $\frac{15}{7}$ (vii) $-\frac{7}{6}$ (viii) $\frac{115}{11}$ (ix) $-\frac{17}{13}$ (x) $\frac{126}{36}$

5. Represent the following decimals in $\frac{p}{q}$ form:

   (i) 2.4 (ii) – 0.32 (iii) 8.14 (iv) $3.\overline{24}$

   (v) 0.415415415...

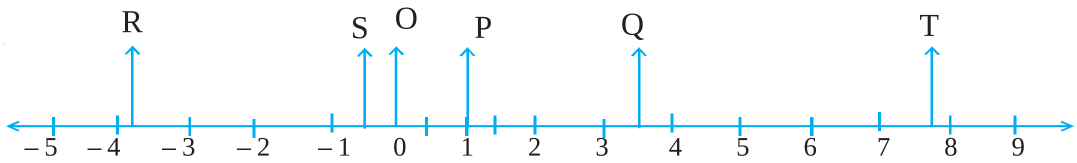
6. Find a rational number betwen the following rational numbers:

   (i) $\frac{3}{4}$ and $\frac{7}{8}$ (ii) – 2 and – 3 (iii) $-\frac{4}{5}$ and $\frac{1}{3}$

7. Find three rational numbers between the following rational numbers:

   (i) $\frac{3}{4}$ and $\frac{-3}{4}$ (ii) 0.27 and 0.28 (iii) 1.32 and 1.34

8. Write the rational numbers corresponding to the points O, P, Q, R, S and T on the number line in the following figure:



9. Find the sum of the following rational numbers:

   (i) $\frac{3}{5}, \frac{-7}{5}$ (ii) $-\frac{7}{9}, \frac{5}{9}$ (iii) $\frac{3}{5}, \frac{7}{3}$ (iv) $\frac{9}{5}, \frac{2}{3}$ (v) $\frac{18}{7}, -\frac{7}{6}$

10. Find the product of the following rationals:

    (i) $\frac{3}{5}, \frac{7}{3}$ (ii) $\frac{19}{5}, \frac{2}{3}$ (iii) $\frac{15}{7}, \frac{-14}{5}$

11. Write an irrational number between the following pairs of numbers:

    (i) 1 and 3 (ii) $\sqrt{3}$ and 3 (iii) $\sqrt{2}$ and $\sqrt{5}$ (iv) $-\sqrt{2}$ and $\sqrt{2}$

12. How many rational numbers and irrational numbers lie between the numbers 2 and 7?

13. Find the approximate value of the following numbers correct to 2 places of decimals:

    (i) 0.338 (ii) 3.924 (iii) 3.14159 (iv) 3.1428

14. Write the value of following correct upto 3 places of decimals:

(i) $\frac{3}{4}$ (ii) $2+\sqrt{2}$ (iii) 1.7326 (iv) 0.9999...

15. Simplify the following as irrational numbers. The first one is done for you.

(i) $12\sqrt{3}+5\sqrt{3}-7\sqrt{3}=\sqrt{3}[12+5-7]=10\sqrt{3}$

(ii) $3\sqrt{2}-2\sqrt{8}+7\sqrt{2}$

(iii) $3\sqrt{2}\times2\sqrt{3}\times5\sqrt{6}$

(iv) $[(\sqrt{8}\times3\sqrt{2})\times6\sqrt{2}]\div36\sqrt{2}$

ANSWERS TO CHECK YOUR PROGRESS

1.1

1. Integers: 4, – 36, – 6

   Rational Numbers: $4,\frac{-3}{4},\frac{5}{6},-36,\frac{12}{7},\frac{-3}{8},\frac{15}{7},-6$

2. (i) $-\frac{7}{4},-\frac{3}{7},-15,0.\frac{5}{17},-\frac{3}{4},-\frac{4}{3}$

   (ii) $-\frac{7}{4},-\frac{3}{7},-15,\frac{5}{17},-\frac{3}{4},-\frac{4}{3}$

   (iii) $-\frac{7}{4},-\frac{3}{7},\frac{5}{17},-\frac{3}{4},-\frac{4}{3}$

   (iv) All are rational numbers.

3. (i) $\frac{16}{3}$, rational (ii) $-\frac{1}{2}$, rational (iii) –21, integer and rational

   (iv) zero, whole number, integer and rational (v) 4, All

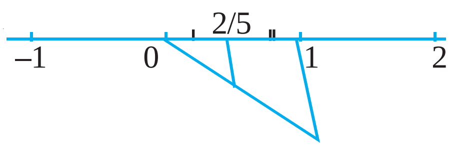
   (vi) $\frac{10}{7}$, rational (vii) $\frac{8}{3}$, rational

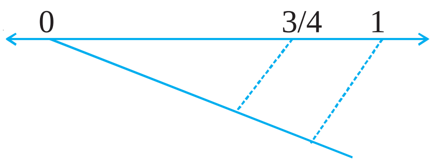
4. (i) 2 (ii) – 8 (iii) 1

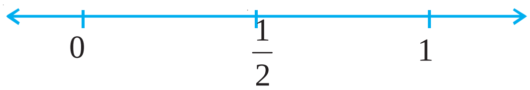
5. $\frac{5}{7}, \frac{-6}{7}, \frac{2\sqrt{3}}{\sqrt{27}}$

6. $-10, \frac{15}{5}, \frac{27}{9}, \frac{-6}{-2}$

7. (i) $\frac{2}{5}=\frac{4}{10}=\frac{6}{15}=\frac{8}{20}$ (ii) $-\frac{5}{6}=-\frac{10}{12}=-\frac{15}{18}=-\frac{20}{24}$ (iii) $\frac{17}{3}=\frac{34}{6}=\frac{51}{9}=\frac{68}{12}$

8. (i) 

(ii) 

(iii) 

9. (a) $\frac{3}{4}>\frac{2}{3}$ (b) $\frac{7}{9}>\frac{3}{5}$ (c) $\frac{-1}{2}>\frac{-2}{3}$ (d) $\frac{5}{11}>\frac{3}{7}$

(e) $\frac{3}{2}>-\frac{7}{6}$

1.2

1. (i) $\frac{9}{7}$ (ii) $-\frac{4}{15}$ (iii) $\frac{1}{2}$ (iv) $\frac{1}{2}$

2. (i) $\frac{19}{6}$ (ii) $\frac{188}{63}$ (iii) $-\frac{11}{35}$

3. (i) $-\frac{53}{48}$ (ii) $\frac{149}{60}$

4. (i) $\frac{2}{5}$ (ii) -4 (iii) $\frac{-3}{56}$

5. (i) $\frac{73}{30}$ (ii) -1

6. (i) $\frac{5}{33}$ (ii) $\frac{9}{35}$ (iii) $\frac{9}{7}$

7. (i) 2 (ii) $\frac{35}{16}$ (iii) $-\frac{10}{3}$

8. (i) $\frac{1}{5}$ (ii) 7

9. $\frac{29}{35}$

10. $\frac{3}{4}$

1.3

1. (i) 0.3875 (ii) 0.48 (iii) 1.5 (iv) 6.25 (v) $1.\overline{4}$

2. (i) $0.\overline{6}$ (ii) $0.\overline{714285}$ (iii) $2.\overline{27}$

3. (a) (i) $\frac{23}{10}$ (ii) $-\frac{78}{25}$ (iii) $-\frac{143}{200}$ (iv) $\frac{4073}{500}$

   (b) (i) $\frac{1}{3}$ (ii) $\frac{113}{33}$ (iii) $-\frac{35}{111}$

1.4

1. (i) $\frac{25}{24}$ (ii) 5.5 (iii) $\frac{-5}{24}$

2. (i) 0.2 and 0.3 (ii) – 0.30, – 0.35

3. (i) 0.271, 0.275, 0, 281, 0.285, 0.291

   (ii) 7.315, 7.320 7.325, 7.330, 7.331

   (iii) 21.75, 22.75, 23.75, 24.75, 25.75

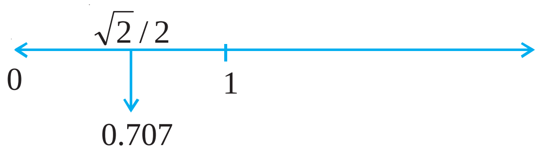
   (iv) 1.0011, 1.0012, 1.0013, 1.0014, 1.0015

   Note: *Can be other answers as well.*

1.5

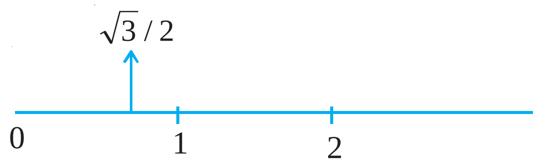
1. 1.414, 1.732, 2.236

2. (i)



(ii)

(iii)



1.6

1. (i) $\sqrt{5}$ (ii) $\sqrt{3}+1$ (iii) $\dfrac{\sqrt{2}+\sqrt{3}}{2}$

2. Infinitely many:

   1.0001, 1.0002, ..., 1.0010, 1.0011,....., 1.0020, 1.0021, ...

1.7

1. (i) 0.778 (ii) 7.326 (iii) 1.012 (iv) 3.143 (v) 1.141

ANASWERS TO TERMINAL EXERCISE

1. Natural: 17,

   Integers but not natural numbers, –3, 0, – 32

   Rationals but not natural numbers: $-3, \dfrac{6}{7}, \dfrac{-3}{8}, 0, -32, \dfrac{3}{14}, \dfrac{11}{6}$

   Irrationals but not rationals: $\sqrt{2},\ \ 2+\sqrt{3}$

2. (i) $-\dfrac{14}{1}$ (ii) $\dfrac{13}{1}$ (iii) $\dfrac{0}{1}$ (iv) $\dfrac{2}{1}$

   (v) $\dfrac{1}{1}$ (vi) $\dfrac{-1}{1}$ (vii) $\dfrac{-25}{1}$

3. $\dfrac{3}{4}, \dfrac{2}{3}, -\dfrac{1}{9}, \dfrac{1}{21}$

4. (i) 0.1375 (ii) 0.32 (iii) 1.75 (iv) 2.5 (v) 2.8

   (vi) 2.142857 (vii) $-1.\overline{166}$ (viii) $10.\overline{45}$ (ix) $-1.\overline{307692}$ (x) 3.5

5. (i) $\frac{12}{5}$ (ii) $\frac{-8}{25}$ (iii) $\frac{407}{50}$ (iv) $\frac{107}{33}$ (v) $\frac{415}{999}$

6. (i) $\frac{13}{16}$ (ii) -2.5 (iii) zero

7. (i) 0.50, 0.25, 0.00 (ii) 0.271, 0.274, 0.277 (iii) 1.325. 1.33, 1.335

8. (i) R: -3.8 (ii) S: -0.5 (iii) O: 0.00 (iv) S: $-0.\overline{33}$ (v) Q: 3.5

   (vi) T: $7.\overline{66}$

9. (i) $-\frac{4}{5}$ (ii) $-\frac{2}{9}$ (iii) $\frac{44}{15}$ (iv) $\frac{37}{15}$ (v) $\frac{59}{42}$

10. (i) $\frac{7}{5}$ (ii) $\frac{38}{15}$ (iii) -6

11. (i) $\sqrt{3}$ (ii) $1+\sqrt{3}$ (iii) $\sqrt{3}$ (iv) $\frac{\sqrt{2}}{2}$

12. Infinitely many

13. (i) 0.34 (ii) 3.92 (iii) 3.14 (iv) 3.14

14. (i) 0.75 (ii) 3.414 (iii) 1.733 (iv) 1.000

15. (ii) $6\sqrt{2}$ (iii) 180 (iv) 2

2

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EXPONENTS AND RADICALS

We have learnt about multiplication of two or more real numbers in the earlier lesson. You can very easily write the following

$$4 \times 4 \times 4 = 64, 11 \times 11 \times 11 \times 11 = 14641 \text{ and}$$

$$2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 256$$

Think of the situation when 13 is to be multiplied 15 times. How difficult is it to write?

$$13 \times 13 \times 13 \times \ldots\ldots\ldots\ldots\ldots 15 \text{ times?}$$

This difficulty can be overcome by the introduction of exponential notation. In this lesson, we shall explain the meaning of this notation, state and prove the laws of exponents and learn to apply these. We shall also learn to express real numbers as product of powers of prime numbers.

In the next part of this lesson, we shall give a meaning to the number $a^{1/q}$ as qth root of *a*. We shall introduce you to radicals, index, radicand etc. Again, we shall learn the laws of radicals and find the simplest form of a radical. We shall learn the meaning of the term “rationalising factor’ and rationalise the denominators of given radicals.

OBJECTIVES

After studying this lesson, you will be able to

- *write a repeated multiplication in exponential notation and vice-versa;*
- *identify the base and exponent of a number written in exponential notation;*
- *express a natural number as a product of powers of prime numbers uniquely;*
- *state the laws of exponents;*
- *explain the meaning of* a^0, a^{-m} *and* $a^{\frac{p}{q}}$;
- *simplify expressions involving exponents, using laws of exponents;*

- *identify radicals from a given set of irrational numbers;*
- *identify index and radicand of a surd;*
- *state the laws of radicals (or surds);*
- *express a given surd in simplest form;*
- *classify similar and non-similar surds;*
- *reduce surds of different orders to those of the same order;*
- *perform the four fundamental operations on surds;*
- *arrange the given surds in ascending/descending order of magnitude;*
- *find a rationalising factor of a given surd;*
- *rationalise the denominator of a given surd of the form* $\frac{1}{a+b\sqrt{x}}$ and $\frac{1}{\sqrt{x}+\sqrt{y}}$,

  where x and y are natural numbers and a and b are integers;
- *simplify expressions involving surds.*

EXPECTED BACKGROUND KNOWLEDGE

- Prime numbers
- Four fundamental operations on numbers
- Rational numbers
- Order relation in numbers.

2.1 EXPONENTIAL NOTATION

Consider the following products:

(i) 7×7 (ii) $3 \times 3 \times 3$ (iii) $6 \times 6 \times 6 \times 6 \times 6$

In (i), 7 is multiplied twice and hence 7×7 is written as 7^2.

In (ii), 3 is multiplied three times and so $3 \times 3 \times 3$ is written as 3^3.

In (iii), 6 is multiplied five times, so $6 \times 6 \times 6 \times 6 \times 6$ is written as 6^5.

7^2 is read as "7 raised to the power 2" or "second power of 7". Here, 7 is called base and 2 is called exponent (or index)

Similarly, 3^3 is read as "3 raised to the power 3"or "third power of 3". Here, 3 is called the base and 3 is called exponent.

Similarly, 6^5 is read as "6 raised to the power 5"or "Fifth power of 6". Again 6 is base and 5 is the exponent (or index).

From the above, we say that

The notation for writing the product of a number by itself several times is called the Exponential Notation or Exponential Form.

Thus, $5 \times 5 \times \ldots .20$ times $= 5^{20}$ and $(-7) \times (-7) \times \ldots .10$ times $= (-7)^{10}$

In 5^{20}, 5 is the base and exponent is 20.

In $(-7)^{10}$, base is -7 and exponent is 10.

Similarly, exponential notation can be used to write precisely the product of a ratioinal number by itself a number of times.

Thus, $$\frac{3}{5} \times \frac{3}{5} \times \ldots\ldots\ldots 16 \text{ times} = \left(\frac{3}{5}\right)^{16}$$

and $$\left(-\frac{1}{3}\right) \times \left(-\frac{1}{3}\right) \times \ldots\ldots\ldots 10 \text{ times} = \left(-\frac{1}{3}\right)^{10}$$

In general, if a is a rational number, multiplied by itself m times, it is written as a^m.

Here again, a is called the base and m is called the exponent

Let us take some examples to illustrate the above discussion:

Example 2.1: Evaluate each of the following:

(i) $\left(\frac{2}{7}\right)^3$ (ii) $\left(-\frac{3}{5}\right)^4$

Solution: (i) $$\left(\frac{2}{7}\right)^3 = \frac{2}{7} \times \frac{2}{7} \times \frac{2}{7} = \frac{(2)^3}{(7)^3} = \frac{8}{343}$$

(ii) $$\left(-\frac{3}{5}\right)^4 = \left(-\frac{3}{5}\right)\left(-\frac{3}{5}\right)\left(-\frac{3}{5}\right)\left(-\frac{3}{5}\right) = \frac{(-3)^4}{(5)^5} = \frac{81}{625}$$

Example 2.2: Write the following in exponential form:

(i) $(-5) \times (-5) \times (-5) \times (-5) \times (-5) \times (-5) \times (-5)$

(ii) $\left(\frac{3}{11}\right) \times \left(\frac{3}{11}\right) \times \left(\frac{3}{11}\right) \times \left(\frac{3}{11}\right)$

Solution: (i) $(-5) \times (-5) \times (-5) \times (-5) \times (-5) \times (-5) \times (-5) = (-5)^7$

(ii) $$\left(\frac{3}{11}\right) \times \left(\frac{3}{11}\right) \times \left(\frac{3}{11}\right) \times \left(\frac{3}{11}\right) = \left(\frac{3}{11}\right)^4$$

Example 2.3: Express each of the following in exponential notation and write the base and exponent in each case.

(i) 4096 (ii) $\frac{125}{729}$ (iii) – 512

Solution: (i) $4096 = 4 \times 4 \times 4 \times 4 \times 4 \times 4$

$= (4)^6$

Alternatively $4096 = (2)^{12}$

Base = 2, exponent = 12

Here, base = 4 and exponent = 6

(ii) $\frac{125}{729} = \frac{5}{9} \times \frac{5}{9} \times \frac{5}{9} = \left(\frac{5}{9}\right)^3$

Here, base = $\left(\frac{5}{9}\right)$ and exponent = 3

(iii) $512 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^9$

Here, base = 2 and exponent = 9

Example 2.4: Simplify the following:

$$\left(\frac{3}{2}\right)^3 \times \left(\frac{4}{3}\right)^4$$

Solution:

$$\left(\frac{3}{2}\right)^3 = \frac{3}{2} \times \frac{3}{2} \times \frac{3}{2} = \frac{3^3}{2^3}$$

Similarly

$$\left(\frac{4}{3}\right)^4 = \frac{4^4}{3^4}$$

$$\left(\frac{3}{2}\right)^3 \times \left(\frac{4}{3}\right)^4 = \frac{3^3}{2^3} \times \frac{4^4}{3^4}$$

$$= \frac{3^3}{8} \times \frac{16 \times 16}{3^4} = \frac{32}{3}$$

Example 2.5: Write the reciprocal of each of the following and express them in exponential form:

(i) 3^5 (ii) $\left(\frac{3}{4}\right)^2$ (iii) $\left(-\frac{5}{6}\right)^9$

Solution: (i) $3^5 = 3 \times 3 \times 3 \times 3 \times 3$

$= 243$

$$\therefore \text{ Reciprocal of } 3^5 = \frac{1}{243} = \left(\frac{1}{3}\right)^5$$

(ii) $$\left(\frac{3}{4}\right)^2 = \frac{3^2}{4^2}$$

$$\therefore \text{ Reciprocal of } \left(\frac{3}{4}\right)^2 = \frac{4^2}{3^2} = \left(\frac{4}{3}\right)^2$$

(iii) $$\left(-\frac{5}{6}\right)^9 = \frac{(-5)^9}{6^9}$$

$$\therefore \text{ Reciprocal of } \left(-\frac{5}{6}\right)^9 = \frac{-6^9}{5^9} = \left(\frac{-6}{5}\right)^9$$

From the above example, we can say that if $\frac{p}{q}$ is any non-zero rational number and m is any positive integer, then the reciprocal of $\left(\frac{p}{q}\right)^m$ is $\left(\frac{q}{p}\right)^m$.

CHECK YOUR PROGRESS 2.1

1. Write the following in exponential form:

   (i) $(-7) \times (-7) \times (-7) \times (-7)$

   (ii) $\left(\frac{3}{4}\right) \times \left(\frac{3}{4}\right) \times \ldots$ 10 times

   (iii) $\left(-\frac{5}{7}\right) \times \left(-\frac{5}{7}\right) \times \ldots$ 20 times

2. Write the base and exponent in each of the following:

(i) $(-3)^5$ (ii) $(7)^4$ (iii) $\left(-\frac{2}{11}\right)^8$

3. Evaluate each of the following

(i) $\left(\frac{3}{7}\right)^4$ (ii) $\left(\frac{-2}{9}\right)^4$ (iii) $\left(-\frac{3}{4}\right)^3$

4. Simplify the following:

(i) $\left(\frac{7}{3}\right)^5 \times \left(\frac{3}{7}\right)^6$

(ii) $\left(-\frac{5}{6}\right)^2 \div \left(-\frac{3}{5}\right)^2$

5. Find the reciprocal of each of the following:

(i) 3^5 (ii) $(-7)^4$ (iii) $\left(-\frac{3}{5}\right)^4$

2.2 PRIME FACTORISATION

Recall that any composite number can be expressed as a product of prime numbers. Let us take the composite numbers 72, 760 and 7623.

(i) $72 = 2 \times 2 \times 2 \times 3 \times 3$

$= 2^3 \times 3^2$

$$\begin{array}{r|l} 2 & 72 \\ \hline 2 & 36 \\ \hline 2 & 18 \\ \hline 3 & 9 \\ \hline & 3 \end{array}$$

(ii) $760 = 2 \times 2 \times 2 \times 5 \times 19$

$= 2^3 \times 5^1 \times 19^1$

$$\begin{array}{r|l} 2 & 760 \\ \hline 2 & 380 \\ \hline 2 & 190 \\ \hline 5 & 95 \\ \hline & 19 \end{array}$$

(iii) $7623 = 3 \times 3 \times 7 \times 11 \times 11$

$= 3^2 \times 7^1 \times 11^2$

$$\begin{array}{r|l} 3 & 7623 \\ \hline 3 & 2541 \\ \hline 7 & 847 \\ \hline 11 & 121 \\ \hline & 11 \end{array}$$

We can see that any natural number, other than 1, can be expressed as a product of powers of prime numbers in a unique manner, apart from the order of occurrence of factors. Let us consider some examples

Example 2.6: Express 24300 in exponential form.

Solution: $24300 = 3 \times 3 \times 3 \times 3 \times 2 \times 2 \times 5 \times 5 \times 3$

$\therefore\ 24300 = 2^2 \times 3^5 \times 5^2$

Example 2.7: Express 98784 in exponential form.

Solution:

$$\begin{array}{r|l} 2 & 98784 \\ \hline 2 & 49392 \\ \hline 2 & 24696 \\ \hline 2 & 12348 \\ \hline 2 & 6174 \\ \hline 3 & 3087 \\ \hline 3 & 1029 \\ \hline 7 & 343 \\ \hline 7 & 49 \\ \hline & 7 \end{array}$$

$\therefore\ 98784 = 2^5 \times 3^2 \times 7^3$

CHECK YOUR PROGRESS 2.2

1. Express each of the following as a product of powers of primes, i.e, in exponential form:

   (i) 429 (ii) 648 (iii) 1512

2. Express each of the following in exponential form:

   (i) 729 (ii) 512 (iii) 2592

   (iv) $\frac{1331}{4096}$ (v) $-\frac{243}{32}$

2.3 LAWS OF EXPONENTS

Consider the following

(i) $3^2 \times 3^3 = (3 \times 3) \times (3 \times 3 \times 3) = (3 \times 3 \times 3 \times 3 \times 3)$

$= 3^5 = 3^{2+3}$

(ii) $(-7)^2 \times (-7)^4 = [(-7) \times (-7)] \times [(-7) \times (-7) \times (-7) \times (-7)]$

$= [\,(-7) \times (-7) \times (-7) \times (-7) \times (-7) \times (-7)]$

$= (-7)^6 = (-7)^{2+4}$

(iii) $\left(\frac{3}{4}\right)^3 \times \left(\frac{3}{4}\right)^4 = \left(\frac{3}{4} \times \frac{3}{4} \times \frac{3}{4}\right) \times \left(\frac{3}{4} \times \frac{3}{4} \times \frac{3}{4} \times \frac{3}{4}\right)$

$$= \left(\frac{3}{4}\times\frac{3}{4}\times\frac{3}{4}\times\frac{3}{4}\times\frac{3}{4}\times\frac{3}{4}\times\frac{3}{4}\right)$$

$$= \left(\frac{3}{4}\right)^7 = \left(\frac{3}{4}\right)^{3+4}$$

(iv) $a^3 \times a^4 = (a \times a \times a) \times (a \times a \times a \times a) = a^7 = a^{3+4}$

From the above examples, we observe that

Law 1: If a is any non-zero rational number and m and n are two positive integers, then

$$a^m \times a^n = a^{m+n}$$

Example 2.8: Evaluate $\left(-\frac{3}{2}\right)^3 \times \left(-\frac{3}{2}\right)^5$.

Solution: Here $a = -\frac{3}{2}$, $m = 3$ and $n = 5$.

$$\therefore \left(-\frac{3}{2}\right)^3 \times \left(-\frac{3}{2}\right)^5 = \left(-\frac{3}{2}\right)^{3+5} = \left(-\frac{3}{2}\right)^8 = \frac{6561}{256}$$

Example 2.9: Find the value of

$$\left(\frac{7}{4}\right)^2 \times \left(\frac{7}{4}\right)^3$$

Solution: As before,

$$\left(\frac{7}{4}\right)^2 \times \left(\frac{7}{4}\right)^3 = \left(\frac{7}{4}\right)^{2+3} = \left(\frac{7}{4}\right)^5 = \frac{16807}{1024}$$

Now study the following:

(i) $7^5 \div 7^3 = \dfrac{7^5}{7^3} = \dfrac{7\times7\times7\times7\times7}{7\times7\times7} = 7\times7 = 7^2 = 7^{5-3}$

(ii) $(-3)^7 \div (-3)^4 = \dfrac{(-3)^7}{(-3)^4} = \dfrac{(-3)\times(-3)\times(-3)\times(-3)\times(-3)\times(-3)\times(-3)}{(-3)\times(-3)\times(-3)\times(-3)}$

$$= (-3)(-3)(-3) = (-3)^3 = (-3)^{7-4}$$

From the above, we can see that

Law 2: If a is any non-zero rational number and m and n are positive integers ($m > n$), then

$$a^m \div a^n = a^{m-n}$$

Example 2.10: Find the value of $\left(\frac{35}{25}\right)^{16} \div \left(\frac{35}{25}\right)^{13}$.

Solution:

$$\left(\frac{35}{25}\right)^{16} \div \left(\frac{35}{25}\right)^{13}$$

$$= \left(\frac{35}{25}\right)^{16-13} = \left(\frac{35}{25}\right)^{3} = \left(\frac{7}{5}\right)^{3} = \frac{343}{125}$$

In Law 2, $m < n \Rightarrow n > m,$

then $$a^m \div a^n = a^{-(n-m)} = \frac{1}{a^{m-n}}$$

Law 3: When $n > m$

$$a^m \div a^n = \frac{1}{a^{m-n}}$$

Example 2.11: Find the value of $\left(\frac{3}{7}\right)^{6} \div \left(\frac{3}{7}\right)^{9}$

Solution: Here $a = \frac{3}{7}$, $m = 6$ and $n = 9$.

$$\therefore \left(\frac{3}{7}\right)^{6} \div \left(\frac{3}{7}\right)^{9} = \left(\frac{3}{7}\right)^{\frac{1}{9-6}}$$

$$= \frac{7^3}{3^3} = \frac{343}{27}$$

Let us consider the following:

(i) $(3^3)^2 = 3^3 \times 3^3 = 3^{3+3} = 3^6 = 3^{3\times 2}$

(ii) $\left[\left(\frac{3}{7}\right)^2\right]^5 = \left(\frac{3}{7}\right)^2 \times \left(\frac{3}{7}\right)^2 \times \left(\frac{3}{7}\right)^2 \times \left(\frac{3}{7}\right)^2 \times \left(\frac{3}{7}\right)^2$

$$\left(\frac{3}{7}\right)^{2+2+2+2+2} = \left(\frac{3}{7}\right)^{10} = \left(\frac{3}{7}\right)^{2\times5}$$

From the above two cases, we can infer the following:

Law 4: If a is any non-zero rational number and m and n are two positive integers, then

$$\left(a^m\right)^n = a^{mn}$$

Let us consider an example.

Example 2.12: Find the value of $\left[\left(\frac{2}{5}\right)^2\right]^3$

Solution: $$\left[\left(\frac{2}{5}\right)^2\right]^3 = \left[\frac{2}{5}\right]^{2\times3} = \left(\frac{2}{5}\right)^6 = \frac{64}{15625}$$

2.3.1 Zero Exponent

Recall that $a^m \div a^n = a^{m-n}$, if $m > n$

$$= \frac{1}{a^{n-m}}, \text{ if } n > m$$

Let us consider the case, when $m = n$

$$\therefore\ a^m \div a^m = a^{m-m}$$

$$\Rightarrow \frac{a^m}{a^m} = a^0$$

$$\Rightarrow 1 = a^0$$

Thus, we have another important law of exponents,.

Law 5: If a is any rational number other than zero, then $a^o = 1$.

Example 2.13: Find the value of

(i) $\left(\frac{2}{7}\right)^0$ (ii) $\left(\frac{-3}{4}\right)^0$

Solution: (i) Using $a^0 = 1$, we get $\left(\frac{2}{7}\right)^0 = 1$

(ii) Again using $a^0 = 1$, we get $\left(\frac{-3}{4}\right)^0 = 1.$

CHECK YOUR PROGRESS 2.3

1. Simplify and express the result in exponential form:

   (i) $(7)^2 \times (7)^3$ (ii) $\left(\frac{3}{4}\right)^3 \times \left(\frac{3}{4}\right)^2$ (iii) $\left(-\frac{7}{8}\right)^1 \times \left(-\frac{7}{8}\right)^2 \times \left(-\frac{7}{8}\right)^3$

2. Simplify and express the result in exponential form:

   (i) $(-7)^9 \div (-7)^7$ (ii) $\left(\frac{3}{4}\right)^8 \div \left(\frac{3}{4}\right)^2$ (iii) $\left(\frac{-7}{3}\right)^{18} \div \left(\frac{-7}{3}\right)^3$

3. Simplify and express the result in exponential form:

   (i) $\left(2^6\right)^3$ (ii) $\left[\left(\frac{3}{4}\right)^3\right]^2$ (iii) $\left[\left(-\frac{5}{9}\right)^3\right]^5$

   (iv) $\left(\frac{11}{3}\right)^5 \times \left(\frac{15}{7}\right)^0$ (v) $\left(-\frac{7}{11}\right)^0 \times \left(-\frac{7}{11}\right)^3$

4. Which of the following statements are true?

   (i) $7^3 \times 7^3 = 7^6$ (ii) $\left(\frac{3}{11}\right)^5 \times \left(\frac{3}{11}\right)^2 = \left(\frac{3}{11}\right)^7$

   (iii) $\left[\left(\frac{4}{9}\right)^5\right]^4 = \left(\frac{4}{9}\right)^9$ (iv) $\left[\left(\frac{3}{19}\right)^6\right]^2 = \left(\frac{3}{19}\right)^8$

   (v) $\left(\frac{3}{11}\right)^0 = 0$ (vi) $\left(-\frac{3}{2}\right)^2 = -\frac{9}{4}$

   (vii) $\left(\frac{8}{15}\right)^5 \times \left(\frac{7}{6}\right)^0 = \left(\frac{8}{15}\right)^5$

2.4 NEGATIVE INTEGERS AS EXPONENTS

i) We know that the reciprocal of 5 is $\frac{1}{5}$. We write it as 5^{-1} and read it as 5 raised to power –1.

ii) The reciprocal of (–7) is $-\frac{1}{7}$. We write it as $(-7)^{-1}$ and read it as (–7) raised to the power –1.

iii) The reciprocal of $5^2 = \frac{1}{5^2}$. We write it as 5^{-2} and read it as '5 raised to the power (–2)'.

From the above all, we get

If a is any non-zero rational number and m is any positive integer, then the reciprocal of a^m $\left(\text{i.e.} \frac{1}{a^m}\right)$ is written as a^{-m} and is read as 'a raised to the power (–m)'. Therefore,

$$\frac{1}{a^m} = a^{-m}$$

Let us consider an example.

Example 2.14: Rewrite each of the following with a positive exponent:

$$\text{(i)} \left(\frac{3}{8}\right)^{-2} \qquad \text{(ii)} \left(-\frac{4}{7}\right)^{-7}$$

Solution:

$$\text{(i)} \left(\frac{3}{8}\right)^{-2} = \frac{1}{\left(\frac{3}{8}\right)^2} = \frac{1}{\frac{3^2}{8^2}} = \frac{8^2}{3^2} = \left(\frac{8}{3}\right)^2$$

$$\text{(ii)} \left(-\frac{4}{7}\right)^{-7} = \frac{1}{\left(-\frac{4}{7}\right)^7} = \frac{7^7}{(-4)^7} = \left(-\frac{7}{4}\right)^7$$

From the above example, we get the following result:

If $\frac{p}{q}$ is any non-zero rational number and m is any positive integer, then

$$\left(\frac{p}{q}\right)^{-m} = \frac{q^m}{p^m} = \left(\frac{q}{p}\right)^m.$$

2.5 LAWS OF EXPONENTS FOR INTEGRAL EXPONENTS

After giving a meaning to negative integers as exponents of non-zero rational numbers, we can see that laws of exponents hold good for negative exponents also.

For example.

(i) $$\left(\frac{3}{5}\right)^{-4} \times \left(\frac{3}{5}\right)^{3} = \frac{1}{\left(\frac{3}{5}\right)^{4}} \times \left(\frac{3}{5}\right)^{3} = \frac{3}{5}^{3-4}$$

(ii) $$\left(-\frac{2}{3}\right)^{-2} \times \left(-\frac{2}{3}\right)^{-3} = \frac{1}{\left(-\frac{2}{3}\right)^{2}} \times \frac{1}{\left(-\frac{2}{3}\right)^{3}} = \frac{1}{\left(-\frac{2}{3}\right)^{2+3}} = \left(-\frac{2}{3}\right)^{-2-3}$$

(iii) $$\left(-\frac{3}{4}\right)^{-3} \div \left(-\frac{3}{4}\right)^{-7} = \frac{1}{\left(-\frac{3}{4}\right)^{3}} \div \frac{1}{\left(-\frac{3}{4}\right)^{7}} = \frac{1}{\left(-\frac{3}{4}\right)^{3}} \times \left(-\frac{3}{4}\right)^{7} = \left(-\frac{3}{4}\right)^{7-3}$$

(iv) $$\left(\left(\frac{2}{7}\right)^{-2}\right)^{3} = \left[\left(\frac{7}{2}\right)^{2}\right]^{3} = \left(\frac{7}{2}\right)^{6} = \left(\frac{2}{7}\right)^{-6} = \left(\frac{2}{7}\right)^{-2\times 3}$$

Thus, from the above results, we find that laws 1 to 5 hold good for negative exponents also.

$\therefore$ For any non-zero rational numbers a and b and any integers m and n,

1. $a^m \times a^n \quad = a^{m+n}$
2. $a^m \div a^n \quad = a^{m-n}$ if $m > n$

   $\qquad\qquad = a^{n-m}$ if $n > m$
3. $(a^m)^n = a^{mn}$
4. $(a \times b)^m = a^m \times b^m$

CHECK YOUR PROGRESS 2.4

1. Express $\left(\frac{-3}{7}\right)^{-2}$ as a rational number of the form $\frac{p}{q}$:

2. Express as a power of rational number with positive exponent:

(i) $\left(\frac{3}{7}\right)^{-4}$ (ii) $12^5 \times 12^{-3}$ (iii) $\left[\left(\frac{3}{13}\right)^{-3}\right]^4$

3. Express as a power of a rational number with negative index:

(i) $\left(\frac{3}{7}\right)^{4}$ (ii) $\left[(7)^2\right]^5$ (iii) $\left[\left(-\frac{3}{4}\right)^{2}\right]^5$

4. Simplify:

(i) $\left(\frac{3}{2}\right)^{-3} \times \left(\frac{3}{2}\right)^{7}$ (ii) $\left(-\frac{2}{3}\right)^{-3} \times \left(-\frac{2}{3}\right)^{4}$ (iii) $\left(-\frac{7}{5}\right)^{-4} \div \left(-\frac{7}{5}\right)^{-7}$

5. Which of the following statements are true?

(i) $a^{-m} \times a^n = a^{-m-n}$

(ii) $(a^{-m})^n = a^{-mn}$

(iii) $a^m \times b^m = (ab)^m$

(iv) $a^m \div b^m = \left(\frac{a}{b}\right)^m$

(v) $a^{-m} \times a^{o} = a^m$

2.6 MEANING OF $a^{p/q}$

You have seen that for all integral values of m and n,

$$a^m \times a^n = a^{m+n}$$

What is the method of defining $a^{1/q}$, if a is positive rational number and q is a natural number.

Consider the multiplication

$$\underbrace{a^{\frac{1}{q}} \times a^{\frac{1}{q}} \times a^{\frac{1}{q}} \ldots\ldots\ldots \times a^{\frac{1}{q}}}_{q \text{ times}} = a^{\frac{1}{q}+\frac{1}{q}+\frac{1}{q}+\ldots\ldots q \text{ times}}$$

$$= a^{\frac{q}{q}} = a$$

In other words, the qth power of $a^{\frac{1}{q}} = a$ or

in other words $a^{\frac{1}{q}}$ is the qth root of a and is written as $\sqrt[q]{a}$.

For example,

$$7^{\frac{1}{4}} \times 7^{\frac{1}{4}} \times 7^{\frac{1}{4}} \times 7^{\frac{1}{4}} = 7^{\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}} = 7^{\frac{4}{4}} = 7^{1} = 7$$

or $7^{\frac{1}{4}}$ is the fourth root of 7 and is written as $\sqrt[4]{7}$,

Let us now define rational powers of a

If a is a positive real number, p is an integer and q is a natural number, then

$$a^{\frac{p}{q}} = \sqrt[q]{a^{p}}$$

We can see that

$$\underbrace{a^{\frac{p}{q}} \times a^{\frac{p}{q}} \times a^{\frac{p}{q}} \ldots\ldots\ldots \times a^{\frac{p}{q}}}_{q \text{ times}} = a^{\frac{p}{q}+\frac{p}{q}+\frac{p}{q}+\ldots\ldots q \text{ times}} = a^{\frac{p}{q}.q} = a^{p}$$

$$\therefore a^{\frac{p}{q}} = \sqrt[q]{a^{p}}$$

$\therefore$ $a^{p/q}$ is the qth root of a^p

Consequently, $7^{\frac{2}{3}}$ is the cube root of 7^2.

Let us now write the laws of exponents for rational exponents:

(i) $a^m \times a^n = a^{m+n}$

(ii) $a^m \div a^n = a^{m-n}$

(iii) $(a^m)^n = a^{mn}$

(iv) $(ab)^m = a^m b^m$

(v) $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$

Let us consider some examples to verify the above laws:

Example 2.15: Find the value of

(i) $(625)^{\frac{1}{4}}$ (ii) $(243)^{\frac{2}{5}}$ (iii) $\left(\frac{16}{81}\right)^{-3/4}$

Solution:

(i) $(625)^{\frac{1}{4}} = (5\times5\times5\times5)^{\frac{1}{4}} = \left(5^4\right)^{\frac{1}{4}} = 5^{4\times\frac{1}{4}} = 5$

(ii) $(243)^{\frac{2}{5}} = (3\times3\times3\times3\times3)^{\frac{2}{5}} = \left(3^5\right)^{\frac{2}{5}} = 3^{5\times\frac{2}{5}} = 3^2 = 9$

(iii) $\left(\frac{16}{81}\right)^{\frac{-3}{4}} = \left(\frac{2\times2\times2\times2}{3\times3\times3\times3}\right)^{\frac{-3}{4}}$

$$= \left[\left(\frac{2}{3}\right)^4\right]^{\frac{-3}{4}} = \left(\frac{2}{3}\right)^{4\times\left(\frac{-3}{4}\right)} = \left(\frac{2}{3}\right)^{-3} = \left(\frac{3}{2}\right)^3 = \frac{27}{8}$$

CHECK YOUR PROGRESS 2.5

1. Simplify each of the following:

   (i) $(16)^{3/4}$ (ii) $\left(\frac{27}{125}\right)^{-\frac{2}{3}}$

2. Simplify each of the following:

   (i) $(625)^{-\frac{1}{4}} \div (25)^{-\frac{1}{2}}$

   (ii) $\left(\frac{7}{8}\right)^{-\frac{1}{4}} \times \left(\frac{7}{8}\right)^{\frac{1}{2}} \times \left(\frac{7}{8}\right)^{\frac{3}{4}}$

   (iii) $\left(\frac{13}{16}\right)^{-\frac{3}{4}} \times \left(\frac{13}{16}\right)^{\frac{1}{4}} \times \left(\frac{13}{16}\right)^{\frac{3}{2}}$

2.7 SURDS

We have read in first lesson that numbers of the type $\sqrt{2}, \sqrt{3}$ and $\sqrt{5}$ are all irrational numbers. We shall now study irrational numbers of a particular type called **radicals** or **surds**.

A surd is defined as a positive irrational number of the type $\sqrt[n]{x}$, where it is not possible to find exactly the nth root of x, where x is a positive rational number.

The number $\sqrt[n]{x}$ is a surd if and only if

(i) it is an irrational number

(ii) it is a root of the positive rational number

2.7.1 Some Terminology

In the surd $\sqrt[n]{x}$, the symbol $\sqrt{}$ is called a **radical sign**. The index 'n' is called the **order** of the surd and *x* is called the **radicand**.

Note: i) When order of the surd is not mentioned, it is taken as 2. For example, order of $\sqrt{7}\ \left(=\sqrt[2]{7}\right)$ is 2.

ii) $\sqrt[3]{8}$ is not a surd as its value can be determined as 2 which is a rational.

iii) $\sqrt{2+\sqrt{2}}$, although an irrational number, is not a surd because it is the square root of an irrational number.

2.8 PURE AND MIXED SURD

i) A surd, with rational factor is 1 only, other factor being rrational is called a **pure surd**.

For example, $\sqrt[5]{16}$ and $\sqrt[3]{50}$ are pure surds.

ii) A surd, having rational factor other than 1 alongwith the irrational factor, is called a mixed surd.

For example, $2\sqrt{3}$ and $3\sqrt[3]{7}$ are mixed surds.

2.9 ORDER OF A SURD

In the surd $5\sqrt[3]{4}$, 5 is called the **co-efficient** of the surd, 3 is the **order** of the surd and 4 is the **radicand**. Let us consider some examples:

Example 2.16: State which of the following are surds?

(i) $\sqrt{49}$ (ii) $\sqrt{96}$ (iii) $\sqrt[3]{81}$ (iv) $\sqrt[3]{256}$

Solution: (i) $\sqrt{49} = 7$, which is a rational number.

$\therefore \sqrt{49}$ is not a surd.

(ii) $\sqrt{96} = \sqrt{4 \times 4 \times 6} = 4\sqrt{6}$

$\therefore \sqrt{96}$ is an irrational number.

$\Rightarrow \sqrt{96}$ is a surd.

(iii) $\sqrt[3]{81} = \sqrt[3]{3 \times 3 \times 3 \times 3} = 3\sqrt[3]{3}$, which is irrational

$\therefore \sqrt[3]{81}$ is a surd.

(iv) $\sqrt[3]{256} = \sqrt[3]{4 \times 4 \times 4 \times 4} = 4\sqrt[3]{4}$

$\therefore \sqrt[3]{256}$ is irrational.

$\Rightarrow \sqrt[3]{256}$ is a surd

$\therefore$ (ii), (iii) and (iv) are surds.

Example 2.17: Find "index" and "radicand" in each of the following:

(i) $\sqrt[5]{117}$ (ii) $\sqrt{162}$ (iii) $\sqrt[4]{213}$ (iv) $\sqrt[4]{214}$

Solution: (i) index is 5 and radicand is 117.

(ii) index is 2 and radicand is 162.

(iii) index is 4 and radicand is 213.

(iv) index is 4 and radicand is 214.

Example 2.18: Identify "pure" and "mixed" surds from the following:

(i) $\sqrt{42}$ (ii) $4\sqrt[3]{18}$ (iii) $2\sqrt[4]{98}$

Solution: (i) $\sqrt{42}$ is a pure surd.

(ii) $4\sqrt[3]{18}$ is a mixed surd.

(iii) $2\sqrt[4]{98}$ is a mixed surd.

2.10 LAWS OF RADICALS

Given below are Laws of Radicals: (without proof):

(i) $\left[\sqrt[n]{a}\right]^n = a$

(ii) $\sqrt[n]{a}\,\sqrt[n]{b} = \sqrt[n]{ab}$

(iii) $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}}$

where a and b are positive rational numbers and n is a positive integer.

Let us take some examples to illustrate.

Example 2.19: Which of the following are surds and which are not? Use laws of radicals to ascertain.

(i) $\sqrt{5}\times\sqrt{80}$ (ii) $2\sqrt{15}\div 4\sqrt{10}$

(iii) $\sqrt[3]{4}\times\sqrt[3]{16}$ (iv) $\sqrt{32}\div\sqrt{27}$

Solution: (i) $\sqrt{5}\times\sqrt{80} = \sqrt{5\times 80} = \sqrt{400} = 20\cdot$

which is a rational number.

$\therefore\ \sqrt{5}\times\sqrt{80}$ is not a surd.

(ii) $2\sqrt{15}\div 4\sqrt{10} = \dfrac{2\sqrt{15}}{4\sqrt{10}} = \dfrac{\sqrt{15}}{2\sqrt{10}}$

$= \dfrac{\sqrt{15}}{\sqrt{2\times 2\times 10}} = \dfrac{\sqrt{15}}{\sqrt{40}} = \sqrt{\dfrac{3}{8}}$, which is irrational.

$\therefore\ 2\sqrt{15}\div 4\sqrt{10}$ is a surd.

(iii) $\sqrt[3]{4}\times\sqrt[3]{16} = \sqrt[3]{64} = 4 \Rightarrow$ It is not a surd.

(iv) $\sqrt{32}\div\sqrt{27} = \dfrac{\sqrt{32}}{\sqrt{27}} = \sqrt{\dfrac{32}{27}}$, which is irrational

$\therefore\ \sqrt{32}\div\sqrt{27}$ is a surd.

CHECK YOUR PROGRESS 2.6

1. For each of the following, write index and the radicand:

   (i) $\sqrt[4]{64}$ (ii) $\sqrt[6]{343}$ (iii) $\sqrt{119}$

2. State which of the following are surds:

(i) $\sqrt[3]{64}$ (ii) $\sqrt[4]{625}$ (iii) $\sqrt[6]{216}$

(iv) $\sqrt{5}\times\sqrt{45}$ (v) $3\sqrt{2}\times5\sqrt{6}$

3. Identify pure and mixed surds out of the following:

(i) $\sqrt{32}$ (ii) $2\sqrt[3]{12}$ (iii) $13\sqrt[3]{91}$ (iv) $\sqrt{35}$

2.11 LAWS OF SURDS

Recall that the surds can be expressed as numbers with fractional exponents. Therefore, laws of indices studied in this lesson before, are applicable to them also. Let us recall them here:

(i) $\sqrt[n]{x}\,.\,\sqrt[n]{y}=\sqrt[n]{xy}$ or $x^{\frac{1}{n}}.y^{\frac{1}{n}}=(xy)^{\frac{1}{n}}$

(ii) $\dfrac{\sqrt[n]{x}}{\sqrt[n]{y}}=\sqrt[n]{\dfrac{x}{y}}$ or $\dfrac{x^{\frac{1}{n}}}{y^{\frac{1}{n}}}=\left(\dfrac{x}{y}\right)^{\frac{1}{n}}$

(iii) $\sqrt[m]{\sqrt[n]{x}}=\sqrt[mn]{x}=\sqrt[n]{\sqrt[m]{x}}$ or $\left(x^{\frac{1}{n}}\right)^{\frac{1}{m}}=x^{\frac{1}{mn}}=\left(x^{\frac{1}{m}}\right)^{\frac{1}{n}}$

(iv) $\sqrt[n]{x^m}=x^{\frac{m}{n}}$ or $(x^m)^{\frac{1}{n}}=x^{\frac{m}{n}}$

(v) $\sqrt[n]{x^p}=\sqrt[mn]{x^{pn}}$ or $(x^p)^{\frac{1}{m}}=x^{\frac{p}{m}}=x^{\frac{pn}{mn}}=(x^{pn})^{\frac{1}{mn}}$

Here, x and y are positive rational numbers and m, n and p are positive integers.

Let us illustrate these laws by examples:

(i) $\sqrt[3]{3}\;\sqrt[3]{8}=3^{\frac{1}{3}}\times8^{\frac{1}{3}}=(24)^{\frac{1}{3}}=\sqrt[3]{24}=\sqrt[3]{3\times8}$

(ii) $\dfrac{(5)^{\frac{1}{3}}}{(9)^{\frac{1}{3}}}=\left(\dfrac{5}{9}\right)^{\frac{1}{3}}=\sqrt[3]{\dfrac{5}{9}}$

(iii) $\sqrt[3]{\sqrt[2]{7}} = \sqrt[3]{7^{\frac{1}{2}}} = \left(7^{\frac{1}{2}}\right)^{\frac{1}{3}} = 7^{\frac{1}{6}} = \sqrt[6]{7} = \sqrt[2\times 3]{7} = \sqrt[2]{\sqrt[3]{7}}$

(iv) $\sqrt[5]{4^3} = \left(4^3\right)^{\frac{1}{5}} = 4^{\frac{3}{5}} = 4^{\frac{9}{15}} = \sqrt[15]{4^9} = \sqrt[3\times 5]{4^{3\times 3}}$

Thus, we see that the above laws of surds are verified.

An important point: The order of a surd can be changed by multiplying the index of the surd and index of the radicand by the same positive number.

For example $\quad \sqrt[3]{2} = \sqrt[6]{2^2} = \sqrt[6]{4}$

and $\quad \sqrt[4]{3} = \sqrt[8]{3^2} = \sqrt[8]{9}$

2.12 SIMILAR (OR LIKE) SURDS

Two surds are said to be **similar**, if they can be reduced to the same irrational factor, without consideration for co-efficient.

For example, $3\sqrt{5}$ and $7\sqrt{5}$ are similar surds. Again consider $\sqrt{75} = 5\sqrt{3}$ and $\sqrt{12} = 2\sqrt{3}$. Now $\sqrt{75}$ and $\sqrt{12}$ are expressed as $5\sqrt{3}$ and $2\sqrt{3}$. Thus, they are similar surds.

2.13 SIMPLEST (LOWEST) FORM OF A SURD

A surd is said to be in its simplest form, if it has

a) smallest possible index of the sign

b) no fraction under radical sign

c) no factor of the form a^n, where a is a positive integer, under the radical sign of index n.

For example, $\sqrt[3]{\dfrac{125}{18}} = \sqrt[3]{\dfrac{125\times 12}{18\times 12}} = \dfrac{5}{6}\sqrt[3]{12}$

Let us take some examples.

Example 2.20: Express each of the following as pure surd in the simplest form:

(i) $2\sqrt{7}$ $\qquad$ (ii) $4\sqrt[4]{7}$ $\qquad$ (iii) $\dfrac{3}{4}\sqrt{32}$

Solution:

(i) $2\sqrt{7}=\sqrt{2^2\times 7}=\sqrt{4\times 7}=\sqrt{28}$, which is a pure surd.

(ii) $4\sqrt[4]{7}=\sqrt[4]{4^4\times 7}=\sqrt[4]{256\times 7}=\sqrt[4]{1792}$, which is a pure surd.

(iii) $\dfrac{3}{4}\sqrt{32}=\sqrt{32\times\dfrac{9}{16}}=\sqrt{18}$, which is a pure surd.

Example 2.21: Express as a mixed surd in the simplest form:

(i) $\sqrt{128}$ (ii) $\sqrt[6]{320}$ (iii) $\sqrt[3]{250}$

Solution:

(i) $\sqrt{128}=\sqrt{64\times 2}=8\sqrt{2}$,

which is a mixed surd.

(ii) $\sqrt[6]{320}=6\sqrt{2\times 2\times 2\times 2\times 2\times 2\times 5}$

$=\sqrt[6]{2^6\times 5}=2\sqrt[6]{5}$, which is a mixed surd.

(iii) $\sqrt[3]{250}=\sqrt[3]{5\times 5\times 5\times 2}=5\sqrt[3]{2}$, which is a mixed surd.

CHECK YOUR PROGRESS 2.7

1. State which of the following are pairs of similar surds:

   (i) $\sqrt{8},\sqrt{32}$ (ii) $5\sqrt{3},6\sqrt{18}$ (iii) $\sqrt{20},\sqrt{125}$

2. Express as a pure surd:

   (i) $7\sqrt{3}$ (ii) $3\sqrt[3]{16}$ (iii) $\dfrac{5}{8}\sqrt{24}$

3. Express as a mixed surd in the simplest form:

   (i) $\sqrt[3]{250}$ (ii) $\sqrt[3]{243}$ (iii) $\sqrt[4]{512}$

2.14 FOUR FUNDAMENTAL OPERATIONS ON SURDS

2.14.1 Addition and Subtraction of Surds

As in rational numbers, surds are added and subtracted in the same way.

For example, $5\sqrt{3}+17\sqrt{3}=(5+17)\sqrt{3}=22\sqrt{3}$

and $12\sqrt{5}-7\sqrt{5}=[12-7]\sqrt{5}=5\sqrt{5}$

For adding and subtracting surds, we first change them to similar surds and then perform the operations.

For example i) $\sqrt{50}+\sqrt{288}$

$=\sqrt{5\times5\times2}+\sqrt{12\times12\times2}$

$=5\sqrt{2}+12\sqrt{2}=\sqrt{2}(5+12)=17\sqrt{2}$

ii) $\sqrt{98}-\sqrt{18}$

$=\sqrt{7\times7\times2}-\sqrt{3\times3\times2}$

$=7\sqrt{2}-3\sqrt{2}=(7-3)\sqrt{2}=4\sqrt{2}$

Example 2.22: Simplify each of the following:

(i) $4\sqrt{6}+2\sqrt{54}$

(ii) $45\sqrt{6}-3\sqrt{216}$

Solution: (i) $4\sqrt{6}+2\sqrt{54}$

$=4\sqrt{6}+2\sqrt{3\times3\times6}$

$=4\sqrt{6}+6\sqrt{6}=10\sqrt{6}$

(ii) $45\sqrt{6}-3\sqrt{216}$

$=45\sqrt{6}-3\sqrt{6\times6\times6}$

$=45\sqrt{6}-18\sqrt{6}$

$=27\sqrt{6}$

Example 2.23: Show that

$$24\sqrt{45}-16\sqrt{20}+\sqrt{245}-47\sqrt{5}=0$$

Solution: $24\sqrt{45}-16\sqrt{20}+\sqrt{245}-47\sqrt{5}$

$$= 24\sqrt{3\times3\times5} - 16\sqrt{2\times2\times5} + \sqrt{7\times7\times5} - 47\sqrt{5}$$

$$= 72\sqrt{5} - 32\sqrt{5} + 7\sqrt{5} - 47\sqrt{5}$$

$$= \sqrt{5}[72 - 32 + 7 - 47]$$

$$= \sqrt{5}\times 0 = 0 = \text{RHS}$$

Example 2.24: Simplify: $2\sqrt[3]{16000} + 8\sqrt[3]{128} - 3\sqrt[3]{54} + \sqrt[4]{32}$

Solution: $2\sqrt[3]{16000} = 2\sqrt[3]{10\times10\times10\times8\times2} = 2\times10\times2\sqrt[3]{2} = 40\sqrt[3]{2}$

$$8\sqrt[3]{128} = 8\sqrt[3]{4\times4\times4\times2} = 32\sqrt[3]{2}$$

$$3\sqrt[3]{54} = 3\sqrt[3]{3\times3\times3\times2} = 9\sqrt[3]{2}$$

$$\sqrt[4]{32} = 2\sqrt[4]{2}$$

$\therefore$ Required expression

$$= 40\sqrt[3]{2} + 32\sqrt[3]{2} - 9\sqrt[3]{2} + 2\sqrt[4]{2}$$
$$= (40 + 32 - 9)\sqrt[3]{2} + 2\sqrt[4]{2}$$
$$= 63\sqrt[3]{2} + 2\sqrt[4]{2}$$

CHECK YOUR PROGRESS 2.8

Simplify each of the following:

1. $\sqrt{175} + \sqrt{112}$
2. $\sqrt{32} + \sqrt{200} + \sqrt{128}$
3. $3\sqrt{50} + 4\sqrt{18}$
4. $\sqrt{108} - \sqrt{75}$
5. $\sqrt[3]{24} + \sqrt[3]{81} - 8\sqrt[3]{3}$
6. $6\sqrt[3]{54} - 2\sqrt[3]{16} + 4\sqrt[3]{128}$
7. $12\sqrt{18} + 6\sqrt{20} - 6\sqrt{147} + 3\sqrt{50} + 8\sqrt{45}$

2.14.2 Multiplication and Division in Surds

Two surds can be multiplied or divided if they are of the same order. We have read that the order of a surd can be changed by multiplying or dividing the index of the surd and index of the radicand by the same positive number. Before multiplying or dividing, we change them to the surds of the same order.

Let us take some examples:

$$\sqrt{3}\times\sqrt{2}=\sqrt{3\times2}=\sqrt{6} \qquad \left[\sqrt{3} \text{ and } \sqrt{2} \text{ are of same order}\right]$$

$$\sqrt{12}\div\sqrt{2}=\frac{\sqrt{12}}{\sqrt{2}}=\sqrt{6}$$

Let us multiply $\sqrt{3}$ and $\sqrt[3]{2}$

$$\sqrt{3}=\sqrt[6]{3^3}=\sqrt[6]{27}$$

$$\sqrt[3]{2}=\sqrt[6]{4}$$

$$\therefore \sqrt{3}\times\sqrt[3]{2}=\sqrt[6]{27}\times\sqrt[6]{4}=\sqrt[6]{108}$$

$$\text{and } \frac{\sqrt{3}}{\sqrt[3]{2}}=\frac{\sqrt[6]{27}}{\sqrt[6]{4}}=\sqrt[6]{\frac{27}{4}}$$

Let us consider an example:

Example 2.25: (i) Multiply $5\sqrt[3]{16}$ and $11\sqrt[3]{40}$.

(ii) Divide $15\sqrt[3]{13}$ by $6\sqrt[6]{5}$.

Solution: (i) $5\sqrt[3]{16}\times 11\sqrt[3]{40}$

$$=5\times11\times\sqrt[3]{2\times2\times2\times2}\times\sqrt[3]{2\times2\times2\times5}$$

$$=55\times2\times2\sqrt[3]{2}\ \sqrt[3]{5}$$

$$=220\sqrt[3]{10}$$

(ii) $$\frac{15\sqrt[3]{13}}{6\sqrt[6]{5}}=\frac{5}{2}\cdot\frac{\sqrt[6]{13^2}}{\sqrt[6]{5}}=\frac{5}{2}\sqrt[6]{\frac{169}{5}}$$

Example 2.26: Simplify and express the result in simplest form:

$$2\sqrt{50}\times\sqrt{32}\times2\sqrt{72}$$

Solution: $2\sqrt{50} = 2\sqrt{5\times5\times2} = 10\sqrt{2}$

$\sqrt{32} = \sqrt{2\times2\times2\times2\times2} = 4\sqrt{2}$

$2\sqrt{72} = 2\times6\sqrt{2} = 12\sqrt{2}$

$\therefore$ Given expression

$= 10\sqrt{2}\times4\sqrt{2}\times12\sqrt{2}$

$= 960\sqrt{2}$

2.15 COMPARISON OF SURDS

To compare two surds, we first change them to surds of the same order and then compare their radicands along with their co-efficients. Let us take some examples:

Example 2.27: Which is greater $\sqrt{\frac{1}{4}}$ or $\sqrt[3]{\frac{1}{3}}$?

Solution: $\sqrt{\frac{1}{4}} = \sqrt[6]{\left(\frac{1}{4}\right)^3} = \sqrt[6]{\frac{1}{64}}$

$\sqrt[3]{\frac{1}{3}} = \sqrt[6]{\frac{1}{9}}$

$\frac{1}{9} > \frac{1}{64} \Rightarrow \sqrt[6]{\frac{1}{9}} > \sqrt[6]{\frac{1}{64}} \Rightarrow \sqrt[3]{\frac{1}{3}} > \sqrt{\frac{1}{4}}$

Example 2.28: Arrange in ascending order: $\sqrt[3]{2}$, $\sqrt{3}$ and $\sqrt[6]{5}$.

Solution: LCM of 2, 3, and 6 is 6.

$\therefore \sqrt[3]{2} = \sqrt[6]{2^2} = \sqrt[6]{4}$

$\sqrt{3} = \sqrt[6]{3^3} = \sqrt[6]{27}$

$\sqrt[6]{5} = \sqrt[6]{5}$

Now $\sqrt[6]{4} < \sqrt[6]{5} < \sqrt[6]{27}$

$\Rightarrow \sqrt[3]{2} < \sqrt[6]{5} < \sqrt{3}$

CHECK YOUR PROGRESS 2.9

1. Multiply $\sqrt[3]{32}$ and $5\sqrt[3]{4}$.

2. Multiply $\sqrt{3}$ and $\sqrt[3]{5}$.

3. Divide $\sqrt[3]{135}$ by $\sqrt[3]{5}$.

4. Divide $2\sqrt{24}$ by $\sqrt[3]{320}$.

5. Which is greater $\sqrt[4]{5}$ or $\sqrt[3]{4}$?

6. Which in smaller: $\sqrt[5]{10}$ or $\sqrt[4]{9}$?

7. Arrange in ascending order:

   $\sqrt[3]{2},\ \sqrt[6]{3},\ \sqrt[3]{4}$

8. Arrange in descending order:

   $\sqrt[3]{2},\ \sqrt[4]{3},\ \sqrt[3]{4}$

2.16 RATIONALISATION OF SURDS

Consider the products:

(i) $3^{\frac{1}{2}} \times 3^{\frac{1}{2}} = 3$

(ii) $5^{\frac{7}{11}} \times 5^{\frac{4}{11}} = 5$

(iii) $7^{\frac{1}{4}} \times 7^{\frac{3}{4}} = 7$

In each of the above three multiplications, we see that on multiplying two surds, we get the result as rational number. In such cases, each surd is called the rationalising factor of the other surd.

(i) $\sqrt{3}$ is a rationalising factor of $\sqrt{3}$ and vice-versa.

(ii) $\sqrt[11]{5^4}$ is a rationalising factor of $\sqrt[11]{5^7}$ and vice-versa.

(iii) $\sqrt[4]{7}$ is a rationalising factor of $\sqrt[4]{7^3}$ and vice-versa.

In other words, the process of converting surds to rational numbers is called rationalisation and two numbers which on multiplication give the rational number is called the rationalisation factor of the other.

For example, the rationalising factor of $\sqrt{x}$ is $\sqrt{x}$, of $\sqrt{3}+\sqrt{2}$ is $\sqrt{3}-\sqrt{2}$.

Note:

(i) The quantities $x-\sqrt{y}$ and $x+\sqrt{y}$ are called conjugate surds. Their sum and product are always rational.

(ii) Rationalisation is usually done of the denominator of an expression involving irrational surds.

Let us consider some examples.

Example 2.29: Find the rationalising factors of $\sqrt{18}$ and $\sqrt{12}$.

Solution: $\sqrt{18} = \sqrt{3\times3\times2} = 3\sqrt{2}$

$\therefore$ Rationalising factor is $\sqrt{2}$.

$\sqrt{12} = \sqrt{2\times2\times3} = 2\sqrt{3}$.

$\therefore$ Rationalising factor is $\sqrt{3}$.

Example 2.30: Rationalise the denominator of $\dfrac{\sqrt{2}+\sqrt{5}}{\sqrt{2}-\sqrt{5}}$.

Solution:

$$\frac{\sqrt{2}+\sqrt{5}}{\sqrt{2}-\sqrt{5}} = \frac{(\sqrt{2}+\sqrt{5})(\sqrt{2}+\sqrt{5})}{(\sqrt{2}-\sqrt{5})(\sqrt{2}+\sqrt{5})} = \frac{(\sqrt{2}+\sqrt{5})^2}{-3}$$

$$= -\frac{7+2\sqrt{10}}{3} = -\frac{7}{3}-\frac{2}{3}\sqrt{10}$$

Example 2.31: Rationalise the denominator of $\dfrac{4+3\sqrt{5}}{4-3\sqrt{5}}$.

Solution:

$$\frac{4+3\sqrt{5}}{4-3\sqrt{5}} = \frac{(4+3\sqrt{5})(4+3\sqrt{5})}{(4-3\sqrt{5})(4+3\sqrt{5})}$$

$$= \frac{16+45+24\sqrt{5}}{16-45} = -\frac{61}{29}-\frac{24}{29}\sqrt{5}$$

Example 2.32: Rationalise the denominator of $\dfrac{1}{\sqrt{3}-\sqrt{2}+1}$.

Solution:

$$\frac{1}{\sqrt{3}-\sqrt{2}+1}=\frac{\left(\sqrt{3}-\sqrt{2}\right)-1}{\left[\left(\sqrt{3}-\sqrt{2}\right)+1\right]\left[\left(\sqrt{3}-\sqrt{2}\right)-1\right]}$$

$$=\frac{\sqrt{3}-\sqrt{2}-1}{\left(\sqrt{3}-\sqrt{2}\right)^2-1}=\frac{\sqrt{3}-\sqrt{2}-1}{4-2\sqrt{6}}$$

$$=\frac{\sqrt{3}-\sqrt{2}-1}{4-2\sqrt{6}}\times\frac{4+2\sqrt{6}}{4+2\sqrt{6}}$$

$$=\frac{4\sqrt{3}-4\sqrt{2}-4+6\sqrt{2}-4\sqrt{3}-2\sqrt{6}}{16-24}$$

$$=-\frac{\sqrt{2}-2-\sqrt{6}}{4}=\frac{\sqrt{6}-\sqrt{2}+2}{4}$$

Example 2.33: If $\dfrac{3+2\sqrt{2}}{3-\sqrt{2}}=a+b\sqrt{2}$, find the values of a and b.

Solution:

$$\frac{3+2\sqrt{2}}{3-\sqrt{2}}=\frac{3+2\sqrt{2}}{3-\sqrt{2}}\times\frac{3+\sqrt{2}}{3+\sqrt{2}}=\frac{9+4+9\sqrt{2}}{9-2}$$

$$=\frac{13+9\sqrt{2}}{7}=\frac{13}{7}+\frac{9}{7}\sqrt{2}=\mathrm{a}+\mathrm{b}\sqrt{2}$$

$$\Rightarrow \mathrm{a}=\frac{13}{7},\qquad \mathrm{b}=\frac{9}{7}$$

CHECK YOUR PROGRESS 2.10

1. Find the rationalising factor of each of the following:

   (i) $\sqrt[3]{49}$ (ii) $\sqrt{2}+1$ (iii) $\sqrt[3]{x^2}+\sqrt[3]{y^2}+\sqrt[3]{xy}$

2. Simplify by rationalising the denominator of each of the following:

   (i) $\dfrac{12}{\sqrt{5}}$ (ii) $\dfrac{2\sqrt{3}}{\sqrt{17}}$ (iii) $\dfrac{\sqrt{11}-\sqrt{5}}{\sqrt{11}+\sqrt{5}}$ (iv) $\dfrac{\sqrt{3}+1}{\sqrt{3}-1}$

3. Simplify: $\dfrac{2+\sqrt{3}}{2-\sqrt{3}}+\dfrac{2-\sqrt{3}}{2+\sqrt{3}}$

4. Rationalise the denominator of $\dfrac{1}{\sqrt{3}-\sqrt{2}-1}$

5. If $a=3+2\sqrt{2}$. Find $a+\dfrac{1}{a}$.

6. If $\dfrac{2+5\sqrt{7}}{2-5\sqrt{7}}=x+\sqrt{7}y$, find x and y.

LET US SUM UP

- $a \times a \times a \times$ ..... m times $= a^m$ is the exponential form, where a is the base and m is the exponent.
- Laws of exponent are:

  (i) $a^m \times a^n = a^{m+n}$ (ii) $a^m \div a^n = a^{m-n}$ (iii) $(ab)^m = a^m b^m$ (iv) $\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$

  (v) $\left(a^m\right)^n = a^{mn}$ (vi) $a^o = 1$ (vii) $a^{-m} = \dfrac{1}{a^m}$

- $a^{\frac{p}{q}} = \sqrt[q]{a^p}$
- An irrational number $\sqrt[n]{x}$ is called a surd, if x is a rational number and nth root of x is not a rational number.
- In $\sqrt[n]{x}$, n is called index and x is called radicand.
- A surd with rational co-efficient (other than 1) is called a mixed surd.
- The order of the surd is the number that indicates the root.
- The order of $\sqrt[n]{x}$ is n
- Laws of radicals (a > 0, b > 0)

  (i) $\left[\sqrt[n]{a}\right]^n = a$ (ii) $\sqrt[n]{a}\times\sqrt[n]{b}=\sqrt[n]{ab}$ (iii) $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}=\sqrt[n]{\dfrac{a}{b}}$

- Operations on surds

$$x^{\frac{1}{n}} \times y^{\frac{1}{n}} = (xy)^{\frac{1}{n}};\quad \left(x^{\frac{1}{n}}\right)^{\frac{1}{m}} = x^{\frac{1}{mn}} = \left(x^{\frac{1}{m}}\right)^{\frac{1}{n}};\quad \frac{x^{\frac{1}{n}}}{y^{\frac{1}{n}}} = \left(\frac{x}{y}\right)^{\frac{1}{n}}$$

$$\left(x^{m}\right)^{\frac{1}{n}} = x^{\frac{m}{n}};\quad \sqrt[m]{x^{a}} = \sqrt[mn]{x^{an}} \text{ or } \left(x^{a}\right)^{\frac{1}{m}} = x^{\frac{a}{m}} = x^{\frac{an}{mn}} = \left(x^{an}\right)^{\frac{1}{mn}}$$

- Surds are similar if they have the same irrational factor.
- Similar surds can be added and subtracted.
- Orders of surds can be changed by multiplying index of the surds and index of the radicand by the same positive number.
- Surds of the same order are multiplied and divided.
- To compare surds, we change surds to surds of the same order. Then they are compared by their radicands alongwith co-efficients.
- If the product of two surds is rational, each is called the rationalising factor of the other.
- $x+\sqrt{y}$ is called rationalising factor of $x-\sqrt{y}$ and vice-versa.

TERMINAL EXERCISE

1. Express the following in exponential form:

   (i) $5 \times 3 \times 5 \times 3 \times 7 \times 7 \times 7 \times 9 \times 9$

   (ii) $\left(\frac{-7}{9}\right)\times\left(\frac{-7}{9}\right)\times\left(\frac{-7}{9}\right)\times\left(\frac{-7}{9}\right)$

2. Simplify the following:

   (i) $\left(-\frac{5}{6}\right)^{3}\times\left(\frac{7}{5}\right)^{2}\times\left(\frac{3}{7}\right)^{3}$

   (ii) $\left(\frac{3}{7}\right)^{2}\times\frac{35}{27}\times\left(-\frac{1}{5}\right)^{2}$

3. Simplify and express the result in exponential form:

   (i) $(10)^{2}\times(6)^{2}\times(5)^{2}$

(ii) $\left(-\frac{37}{19}\right)^{20} \div \left(-\frac{37}{19}\right)^{20}$

(iii) $\left[\left(\frac{3}{13}\right)^{3}\right]^{5}$

4. Simplify each of the following:

   (i) $3^o + 7^o + 37^o - 3$ (ii) $(7^o + 3^o)(7^o - 3^o)$

5. Simplify the following:

   (i) $(32)^{12} \div (32)^{-6}$ (ii) $(111)^{6} \times (111)^{-5}$ (iii) $\left(-\frac{2}{9}\right)^{-3} \times \left(-\frac{2}{9}\right)^{5}$

6. Find x so that $\left(\frac{3}{7}\right)^{-3} \times \left(\frac{3}{7}\right)^{11} = \left(\frac{3}{7}\right)^{x}$

7. Find x so that $\left(\frac{3}{13}\right)^{-2} \times \left(\frac{3}{13}\right)^{-9} = \left(\frac{3}{13}\right)^{2x+1}$

8. Express as a product of primes and write the answers of each of the following in exponential form:

   (i) 6480000 (ii) 172872 (iii) 11863800

9. The star sirus is about 8.1×10^{13} km from the earth. Assuming that the light travels at 3.0×10^{5} km per second, find how long light from sirus takes to reach earth.

10. State which of the following are surds:

    (i) $\sqrt{\frac{36}{289}}$ (ii) $\sqrt[9]{729}$ (iii) $\sqrt[3]{\sqrt{5}+1}$ (iv) $\sqrt[4]{3125}$

11. Express as a pure surd:

    (i) $3\sqrt[2]{3}$ (ii) $5\sqrt[3]{4}$ (iii) $5\sqrt[5]{2}$

12. Express as a mixed surd in simplest form:

    (i) $\sqrt[4]{405}$ (ii) $\sqrt[5]{320}$ (iii) $\sqrt[3]{128}$

13. Which of the following are pairs of similar surds?

    (i) $\sqrt{112}, \sqrt{343}$ (ii) $\sqrt[3]{625}, \sqrt[3]{3125 \times 25}$ (iii) $\sqrt[6]{216}, \sqrt{250}$

14.Simplify each of the following:

(i) $4\sqrt{48}-\frac{5}{2}\sqrt{\frac{1}{3}}+6\sqrt{3}$

(ii) $\sqrt{63}+\sqrt{28}-\sqrt{175}$

(iii) $\sqrt{8}+\sqrt{128}-\sqrt{50}$

15. Which is greater?

(i) $\sqrt{2}$ or $\sqrt[3]{3}$ (ii) $\sqrt[3]{6}$ or $\sqrt[4]{8}$

16. Arrange in descending order:

(i) $\sqrt{3},\sqrt[3]{4},\sqrt[4]{5}$ (ii) $\sqrt{2},\sqrt{3},\sqrt[3]{4}$

17. Arrange in ascending order:

$\sqrt[3]{16},\sqrt{12},\sqrt[6]{320}$

18. Simplify by rationalising the denominator:

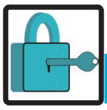
(i) $\frac{3}{\sqrt{6}-\sqrt{7}}$ (ii) $\frac{12}{\sqrt{7}-\sqrt{3}}$ (iii) $\frac{\sqrt{5}-2}{\sqrt{5}+2}$

19. Simplify each of the following by rationalising the denominator:

(i) $\frac{1}{1+\sqrt{2}-\sqrt{3}}$ (ii) $\frac{1}{\sqrt{7}+\sqrt{5}-\sqrt{12}}$

20. If $\frac{5+2\sqrt{3}}{7+4\sqrt{3}}=a+b\sqrt{3},$ find the values of a and b, where a and b are rational numbers.

21. If $x=7+4\sqrt{3}$, find the value of $x+\frac{1}{x}$.



ANSWERS TO CHECK YOUR PROGRESS

2.1

1. (i) $(-7)^4$ (ii) $\left(\frac{3}{4}\right)^{10}$ (iii) $\left(\frac{-5}{7}\right)^{20}$

2.

	Base	Exponent
(i)	-3	5
(ii)	7	4
(iii)	$-\dfrac{2}{11}$	8

3. (i) $\dfrac{81}{2401}$ (ii) $\dfrac{16}{6561}$ (iii) $-\dfrac{27}{64}$

4. (i) $\dfrac{3}{7}$ (ii) $\dfrac{625}{324}$

5. (i) $\left(\dfrac{1}{3}\right)^5$ (ii) $\left(-\dfrac{1}{7}\right)^4$ (iii) $\left(-\dfrac{5}{3}\right)^4$

2.2

1. (i) $3^1 \times 11^1 \times 13^1$ (ii) $2^3 \times 3^4$ (iii) $2^3 \times 3^3 \times 7^1$

2. (i) 3^6 (ii) 2^9 (iii) $2^5 \times 3^4$

   (iv) $\dfrac{11^3}{2^{12}}$ (v) $\dfrac{(-7)^3}{2^5}$

2.3

1. (i) $(7)^5$ (ii) $\left(\dfrac{3}{4}\right)^5$ (iii) $\left(-\dfrac{7}{8}\right)^6$

2. (i) $(-7)^2$ (ii) $\left(\dfrac{3}{4}\right)^6$ (iii) $\left(-\dfrac{7}{8}\right)^{15}$

3. (i) 2^{18} (ii) $\left(\dfrac{3}{4}\right)^6$ (iii) $\left(-\dfrac{5}{9}\right)^{15}$

   (iv) $\left(\dfrac{11}{3}\right)^5$ (v) $\left(-\dfrac{7}{11}\right)^3$

4. True: (i), (ii), (vii)

   False: (iii), (iv), (v), (vi)

2.4

1. $\frac{49}{9}$

2. (i) $\left(\frac{7}{3}\right)^4$ (ii) 12^2 (iii) $\left(\frac{13}{3}\right)^{12}$

3. (i) $\left(\frac{7}{3}\right)^{-4}$ (ii) $\left(\frac{1}{7}\right)^{-10}$ (iii) $\left(-\frac{4}{3}\right)^{-10}$

4. (i) $\frac{81}{16}$ (ii) $-\frac{2}{3}$ (iii) $-\frac{343}{125}$

5. True: (ii), (iii), (iv)

2.5

1. (i) 8 (ii) $\frac{25}{9}$

2. (i) 1 (ii) $\frac{7}{8}$ (iii) $\frac{13}{16}$

2.6

1. (i) 4, 64 (ii) 6, 343 (iii) 2, 119

2. (iii), (iv)

3. Pure: (i), (iv)

   Mixed: (ii), (iii)

2.7

1. (i), (iii)

2. (i) $\sqrt{147}$ (ii) $\sqrt[3]{432}$ (iii) $\sqrt{\frac{75}{8}}$

3. (i) $5\sqrt[3]{2}$ (ii) $3\sqrt[3]{9}$ (iii) $4\sqrt[4]{2}$

2.8

1. $9\sqrt{7}$ 2. $22\sqrt{2}$ 3. $27\sqrt{2}$ 4. $\sqrt{3}$

5. $-3\sqrt{3}$ 6. $30\sqrt[3]{2}$ 7. $51\sqrt{2}+36\sqrt{5}-42\sqrt{3}$

2.9

1. $20\sqrt[3]{2}$ 2. $3\sqrt[3]{5}$ 3. 3 4. $\sqrt[6]{\dfrac{216}{25}}$

5. $\sqrt[3]{4}$ 6. $\sqrt[4]{9}$ 7. $\sqrt[6]{3},\sqrt[3]{2},\sqrt[3]{4}$ 8. $\sqrt[3]{4},\sqrt[4]{3},\sqrt[3]{2}$

2.10

1. (i) $\sqrt[3]{7}$ (ii) $\sqrt{2}-1$ (iii) $\sqrt[3]{x}-\sqrt[3]{y}$

2. (i) $\dfrac{12}{5}\sqrt{5}$ (ii) $\dfrac{2\sqrt{51}}{17}$ (iii) $\dfrac{8}{3}-\dfrac{\sqrt{55}}{3}$ (iv) $2+\sqrt{3}$

3. 14

4. $-\dfrac{1}{4}\left[2+\sqrt{6}+\sqrt{2}\right]$

5. 6

6. $-\dfrac{179}{171}-\dfrac{20\sqrt{7}}{171}$

ANSWERS TO TERMINAL EXERCISE

1. (i) $5^2 \times 3^2 \times 7^3 \times 9^2$ (ii) $\left(-\dfrac{7}{9}\right)^4$

2. (i) $-\dfrac{5}{56}$ (ii) $\dfrac{1}{105}$

3. (i) $2^4 \times 3^2 \times 5^4$ (ii) 1 (iii) $\left(\dfrac{3}{13}\right)^{15}$

4. (i) zero (ii) zero

5. (i) $(32)^{18}$ (ii) 111 (iii) $\left(\dfrac{2}{9}\right)^2$

6. $x = 8$

7. $x = -6$

8. $2^7 \times 3^4 \times 5^4$

9. $3^3 \times 10^7$ seconds

10. (ii), (iii), (iv)

11. (i) $\sqrt[2]{27}$ (ii) $\sqrt[3]{500}$ (iii) $\sqrt[5]{6250}$

12. (i) $3\sqrt[4]{5}$ (ii) $2\sqrt[5]{10}$ (iii) $4\sqrt[3]{2}$

13. (i), (ii)

14. (i) $\frac{127}{6}\sqrt{3}$ (ii) zero (iii) $5\sqrt{2}$

15. (i) $\sqrt[3]{3}$ (ii) $\sqrt[3]{6}$

16. (i) $\sqrt{3}, \sqrt[3]{4}, \sqrt[4]{5}$ (ii) $\sqrt{3}, \sqrt[3]{4}, \sqrt{2}$

17. $\sqrt[3]{16}, \sqrt[6]{320}, \sqrt{12}$

18. (i) $-3\left(\sqrt{6}+\sqrt{7}\right)$ (ii) $3\left(\sqrt{7}+\sqrt{3}\right)$ (ii) $9-4\sqrt{5}$

19. (i) $\frac{2+\sqrt{2}+\sqrt{6}}{4}$ (ii) $\frac{7\sqrt{5}+5\sqrt{7}+2\sqrt{105}}{70}$

20. $a = 11$, $b = -6$

21. 14

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3

ALGEBRAIC EXPRESSIONS AND POLYNOMIALS

So far, you had been using arithmetical numbers, which included natural numbers, whole numbers, fractional numbers, etc. and fundamental operations on those numbers. In this lesson, we shall introduce algebraic numbers and some other basic concepts of algebra like constants, variables, algebraic expressions, special algebraic expressions, called polynomials and four fundamental operations on them.

OBJECTIVES

After studying this lesson, you will be able to

- *identify variables and constants in an expression;*
- *cite examples of algebraic expressions and their terms;*
- *understand and identify a polynomial as a special case of an algebraic expression;*
- *cite examples of types of polynomials in one and two variables;*
- *identify like and unlike terms of polynomials;*
- *determine degree of a polynomial;*
- *find the value of a polynomial for given value(s) of variable(s), including zeros of a polynomial;*
- *perform four fundamental operations on polynomials.*

EXPECTED BACKGROUND KNOWLEDGE

- Knowledge of number systems and four fundamental operations.
- Knowledge of other elementary concepts of mathematics at primary and upper primary levels.

3.1 INTRODUCTION TO ALGEBRA

You are already familiar with numbers 0, 1, 2, 3, ..., $\frac{1}{2}, \frac{3}{4}, \ldots, \sqrt{2}, \ldots$ etc. and operations of addition (+), subtraction (–), multiplication (×) and division (÷) on these numbers. Sometimes, letters called **literal numbers**, are also used as symbols to represent numbers. Suppose we want to say "The cost of one book is twenty rupees".

In arithmetic, we write : The cost of one book = ₹ 20

In algebra, we put it as: the cost of one book in rupees is x. Thus x stands for a number.

Similarly, a, b, c, x, y, z, etc. can stand for number of chairs, tables, monkeys, dogs, cows, trees, etc. The use of letters help us to think in more general terms.

Let us consider an example, you know that if the side of a square is 3 units, its perimeter is 4 × 3 units. In algebra, we may express this as

$$p = 4\,s$$

where p stands for the number of units of perimeter and s those of a side of the square.

On comparing the language of arithmetic and the language of algebra we find that the language of algebra is

(a) more precise than that of arithmetic.

(b) more general than that of arithmetic.

(c) easier to understand and makes solutions of problems easier.

A few more examples in comparative form would confirm our conclusions drawn above:

Verbal statement	Algebraic statement
(i) A number increased by 3 gives 8	$a + 3 = 8$
(ii) A number increased by itself gives 12	$x + x = 12$, written as $2x = 12$
(iii) Distance = speed × time	$d = s \times t$, written as $d = st$
(iv) A number, when multiplied by itself and added to 5 gives 9	$b \times b + 5 = 9$, written as $b^2 + 5 = 9$
(v) The product of two successive natural numbers is 30	$y \times (y + 1) = 30$, wrtten as $y\,(y + 1) = 30$, where y is a natural number.

Since literal numbers are used to represent numbers of arithmetic, symbols of operation +, –, × and ÷ have the same meaning in algebra as in arithmetic. Multiplication symbols in algebra are often omitted. Thus for 5 × a we write 5a and for a × b we write ab.

3.2 VARIABLES AND CONSTANTS

Consider the months — January, February, March, ..., December of the year 2009. If we represent ‘the year 2009’ by a and ‘a month’ by x we find that in this situation ‘a’ (year 2009) is a fixed entity whereas x can be any one of January, February, March, ..., December. Thus, x is not fixed. It varies. We say that in this case ‘a’ is a **constant** and ‘x’ is a **variable**.

Similarly, when we consider students of class X and represent class X by, say, b and a student by, say, y; we find that in this case b (class X) is fixed and so b is a constant and y (a student) is a variable as it can be any one student of class X.

Let us consider another situation. If a student stays in a hostel, he will have to pay fixed room rent, say, ₹ 1000. The cost of food, say ₹ 100 per day, depends on the number of days he takes food there. In this case room rent is constant and the number of days, he takes food there, is variable.

Now think of the numbers.

$$4, -14, \sqrt{2}, \frac{\sqrt{3}}{2}, -\frac{4}{15}, 3x, \frac{21}{8}y, \sqrt{2}z$$

You know that $4, -14, \sqrt{2}, \frac{\sqrt{3}}{2}$, and $-\frac{4}{15}$ are real numbers, each of which has a fixed value while $3x, \frac{21}{8}y$ and $\sqrt{2}z$ contain unknown x, y and z respectively and therefore do not have fixed values like 4, –14, etc. Their values depend on x, y and z respectively. Therefore, x, y and z are variables.

Thus, *a variable is literal number which can have different values whereas a constant has a fixed value.*

In algebra, we usually denote constants by a, b, c and variables x, y, z. However, the context will make it clear whether a literal number has denoted a constant or a variable.

3.3 ALBEGRAIC EXPRESSIONS AND POLYNOMIALS

Expressions, involving arithmetical numbers, variables and symbols of operations are called algebraic expressions. Thus, $3+8, 8x+4, 5y, 7x-2y+6, \frac{1}{\sqrt{2x}}, \frac{x}{\sqrt{y-2}}, \frac{ax+by+cz}{x+y+z}$ are all algebraic expressions. You may note that 3 + 8 is both an arithmetic as well as algebraic expression.

An algebraic expression is a combination of numbers, variables and arithmetical operations.

One or more signs + or – separates an algebraic expression into several parts. Each part along with its sign is called a **term** of the expression. Often, the plus sign of the first term is omitted in writing an algebraic expression. For example, we write $x - 5y + 4$ instead of writing $+ x - 5y + 4$. Here x, $-5y$ and 4 are the three terms of the expression.

In $\frac{1}{3}xy$, $\frac{1}{3}$ is called the numerical coefficient of the term and also of xy. **coefficient** of x is $\frac{1}{3}y$ and that of y is $\frac{1}{3}x$. When the numerical coefficient of a term is $+1$ or -1, the '1' is usually omitted in writing. Thus, numerical coefficent of a term, say, x^2y is $+1$ and that of $-x^2y$ is -1.

An algebraic expression, in which variable(s) does (do) not occur in the denominator, exponents of variable(s) are whole numbers and numerical coefficients of various terms are real numbers, is called a polynomial.

In other words,

(i) No term of a polynomial has a variable in the denominator;

(ii) In each term of a polynomial, the exponents of the variable(s) are non-negative integers; and

(iii) Numerical coefficient of each term is a real number.

Thus, for example, 5, $3x - y$, $\frac{1}{3}a - b + \frac{7}{2}$ and $\frac{1}{4}x^3 - 2y^2 + xy - 8$ are all polynomials whereas $x^3 - \frac{1}{x}$, $\sqrt{x+y}$ and $x^{\frac{2}{3}} + 5$ are not polynomials.

$x^2 + 8$ is a polynomial in one variable x and $2x^2 + y^3$ is a polynomial in two variables x and y. In this lesson, we shall restrict our discussion of polynomials including two variables only.

General form of a polynomial in one variable x is:

$$a_0 + a_1x + a_2x^2 + ...+a_nx^n$$

where coefficients $a_0, a_1, a_2, ...a_n$ are real numbers, x is a variable and n is a whole number. $a_0, a_1x, a_2x^2, ..., a_nx^n$ are $(n + 1)$ terms of the polynomial.

An algebraic expression or a polynomial, consisting of only one term, is called a **monomial**.

Thus, -2, $3y$, $-5x^2$, xy, $\frac{1}{2}x^2y^3$ are all monomials.

An algebraic expression or a polynomial, consisting of only two terms, is called a **binomial**. Thus, $5 + x$, $y^2 - 8x$, $x^3 - 1$ are all bionomials.

An algebraic expression or a polynomial, consisting of only three terms, is called a **trinomial**. Thus $x + y + 1$, $x^2 + 3x + 2$, $x^2 + 2xy + y^2$ are all trinomials.

The terms of a polynomial, having the same variable(s) and the same exponents of the variable(s), are called like terms.

For example, in the expression

$$3xy + 9x + 8xy - 7x + 2x^2$$

the terms 3xy and 8 xy are like terms; also 9x and –7x are **like terms** whereas 9x and $2x^2$ are not like terms. Terms that are not like, are called **unlike terms**. In the above expression 3xy and –7x are also unlike terms.

Note that arithmetical numbers are like terms. For example, in the polynomials $x^2 + 2x + 3$ and $x^3 - 5$, the terms 3 and – 5 are regrded as like terms since $3 = 3x^0$ and $-5 = -5x^0$.

The terms of the expression

$$2x^2 - 3xy + 9y^2 - 7y + 8$$

are all unlike, i.e., there are no two like terms in this expression.

Example 3.1: Write the variables and constants in $2x^2y + 5$.

Solution: Variables : x and y

Constants: 2 and 5

Example 3.2: In $8x^2y^3$, write the coefficient of

(i) x^2y^3 (ii) x^2 (iii) y^3

Solution: (i) $8x^2y^3 = 8 \times (x^2y^3)$

$\therefore$ Coefficient of x^2y^3 is 8

(ii) $8x^2y^3 = 8y^3 \times (x^2)$

$\therefore$ Coefficient of x^2 is $8y^3$.

(iii) $8x^2y^3 = 8x^2 \times (y^3)$

$\therefore$ Coefficient of y^3 is $8x^2$.

Example 3.3: Write the terms of expression

$$3x^2y - \frac{5}{2}x - \frac{1}{3}y + 2$$

Solution: The terms of the given expression are

$$3x^2y,\ -\frac{5}{2}x,\ -\frac{1}{3}y,\ 2$$

Example 3.4: Which of the following algebraic expressions are polynomials?

(i) $\frac{1}{2}+x^3-2x^2+\sqrt{6}x$ (ii) $x+\frac{1}{x}$

(iii) $2x^2+3x-5\sqrt{x}+6$ (iv) $5-x-x^2-x^3$

Solution: (i) and (iv) are polynomials.

In (ii), second term is $\frac{1}{x}=x^{-1}$. Since second term contains negative exponent of the variable, the expression is not a polynomial.

In (iii), third term is $-5\sqrt{x}=-5x^{\frac{1}{2}}$. Since third term contains fractional exponent of the variable, the expression is not a polynomial.

Example 3.5: Write like terms, if any, in each of the following expressions:

(i) $x+y+2$ (ii) $x^2-2y-\frac{1}{2}x^2+\sqrt{3}y-8$

(iii) $1-2xy+2x^2y-2xy^2+5x^2y^2$ (iv) $\frac{2}{\sqrt{3}}y-\frac{1}{3}z+\frac{\sqrt{5}}{3}y+\frac{1}{3}$

Solution: (i) There are no like terms in the expression.

(ii) x^2 and $-\frac{1}{2}x^2$ are like terms, also $-2y$ and $\sqrt{3}\,y$ are like terms

(iii) There are no like terms in the expression.

(iv) $\frac{2}{\sqrt{3}}y$ and $\frac{\sqrt{5}}{3}y$ are like terms

CHECK YOUR PROGRESS 3.1

1. Write the variables and constants in each of the following:

(i) $1+y$ (ii) $\frac{2}{3}x+\frac{1}{3}y+7$ (iii) $\frac{4}{5}x^2y^3$

(iv) $\frac{2}{5}xy^5+\frac{1}{2}$ (v) $2x^2+y^2-8$ (vi) $x+\frac{1}{x}$

2. In $2x^2y$, write the coefficient of

   (i) x^2y (ii) x^2 (iii) y

3. Using variables and operation symbols, express each of the following verbal statements as algebraic statements:

   (i) three less than a number equals fifteen.

   (ii) A number increased by five gives twenty-two.

4. Write the terms of each of the following expressions:

   (i) $2 + abc$ (ii) $a + b + c + 2$ (iii) $x^2y - 2xy^2 - \frac{1}{2}$

   (iv) $\frac{1}{8}x^3y^2$

5. Identify like terms, if any, in each of the following expressions:

   (i) $-xy^2 + x^2y + y^2 + \frac{1}{3}y^2x$ (ii) $6a + 6b - 3ab + \frac{1}{4}a^2b + ab$

   (iii) $ax^2 + by^2 + 2c - a^2x - b^2y - \frac{1}{3}c^2$

6. Which of the following algebraic expressions are polynomials?

   (i) $\frac{1}{3}x^3 + 1$ (ii) $5^2 - y^2 - 2$ (iii) $4x^{-3} + 3y$

   (iv) $5\sqrt{x+y} + 6$ (v) $3x^2 - \sqrt{2}\,y^2$ (vi) $y^2 - \frac{1}{y^2} + 4$

7. Identify each of the following as a monomial, binomial or a trinomial:

   (i) $x^3 + 3$ (ii) $\frac{1}{3}x^3y^3$ (iii) $2y^2 + 3yz + z^2$

   (iv) $5 - xy - 3x^2y^2$ (v) $7 - 4x^2y^2$ (vi) $-8x^3y^3$

3.4 DEGREE OF A POLYNOMIAL

The sum of the exponents of the variables in a term is called the **degree** of that term. For example, the degree of $\frac{1}{2}x^2y$ is 3 since the sum of the exponents of x and y is 2 + 1, i.e., 3. Similarly, the degree of the term $2x^5$ is 5. The degree of a non-zero constant, say, 3 is 0 since it can be written as $3 = 3 \times 1 = 3 \times x^0$, as $x^0 = 1$.

A polynomial has a number of terms separated by the signs + or –. **The degree of a polynomial is the same as the degree of its term or terms having the highest degree and non-zero coefficient.**

For example, consider the polynomial

$$3x^4y^3 + 7\,xy^5 - 5x^3y^2 + 6xy$$

It has terms of degrees 7, 6, 5, and 2 respectively, of which 7 is the highest. Hence, the degree of this polynomial is 7.

A polynomial of degree 2 is also called a **quadratic polynomial**. For example, $3 - 5x + 4x^2$ and $x^2 + xy + y^2$ are quadratic polynomials.

Note that **the degree of a non-zero constant polynomial is taken as zero.**

When all the coefficients of variable(s) in the terms of a polynomial are zeros, the polynomial is called a **zero polynomial. The degree of a zero polynomial is not defined.**

3.5 EVALUATION OF POLYNOMIALS

We can evaluate a polynomial for given value of the variable occuring in it. Let us understand the steps involved in evaluation of the polynomial $3x^2 - x + 2$ for $x = 2$. Note that we restrict ourselves to polynomials in one variable.

Step 1: Substitute given value(s) in place of the variable(s).

Here, when $x = 2$, we get $3 \times (2)^2 - 2 + 2$

Step 2: Simplify the numerical expression obtained in Step 1.

$3 \times (2)^2 - 2 + 2 = 3 \times 4 = 12$

Therefore, when $x = 2$, we get $3x^2 - x + 2 = 12$

Let us consider another example.

Example 3.6: Evaluate

(i) $1 - x^5 + 2x^6 + 7\,x$ for $x = \frac{1}{2}$

(ii) $5x^3 + 3x^2 - 4x - 4$ for $x = 1$

Solution: (i) For $x = \frac{1}{2}$, the value of the given polynomial is:

$$= 1 - \left(\frac{1}{2}\right)^5 + 2\left(\frac{1}{2}\right)^6 + 7 \times \frac{1}{2}$$

$$= 1 - \frac{1}{32} + \frac{1}{32} + \frac{7}{2}$$

$$= \frac{9}{2} = 4\frac{1}{2}$$

(ii) For $x = 1$, the value of the given polynomial is:

$5 \times (1)^3 + 3 \times (1)^2 - 4 \times 1 - 4$

$= 5 + 3 - 4 - 4 = 0$

3.6 ZERO OF A POLYNOMIAL

The value(s) of the variable for which the value of a polynomial in one variable is zero is (are) called **zero(s) of the polynomial**. In Example 3.6(ii) above, the value of the polynomial $5x^3 + 3x^2 - 4x - 4$ for $x = 1$ is zero. Therefore, we say that $x = 1$ is a zero of the polynomial $5x^3 + 3x^2 - 4x - 4$.

Let us consider another example.

Example 3.7: Determine whether given value is a zero of the given polynomial:

(i) $x^3 + 3x^2 + 3x + 2;\ x = -1$

(ii) $x^4 - 4x^3 + 6x^2 - 4x + 1;\ x = 1$

Solution: (i) For $x = -1$, the value of the given polynomial is

$(-1)^3 + 3 \times (-1)^2 + 3 \times (-1) + 2$

$= -1 + 3 - 3 + 2$

$= 1\ (\neq 0)$

Hence, $x = -1$ is not a zero of the given polynomial.

(ii) For $x = 1$, the value of the given polynomial is

$(1)^4 - 4 \times (1)^3 + 6 \times (1)^2 - 4 \times 1 + 1$

$= 1 - 4 + 6 - 4 + 1$

$= 0$

Hence, $x = 1$ is a zero of the given polynomial.

CHECK YOUR PROGRESS 3.2

1. Write the degree of each of the following monomials:

(i) $\frac{18}{5}x^7$ (ii) $\frac{7}{8}y^3$ (iii) $10x$ (iv) 27

2. Rewrite the following monomials in increasing order of their degrees:

   $-3x^6$, $\frac{2}{9}x^2$, $9x$, $-25x^3$, 2.5

3. Determine the degree of each of the following polynomials:

   (i) $5x^6y^4 + 1$ (ii) $10^5 + xy^3$ (iii) $x^2 + y^2$ (iv) $x^2y + xy^2 - 3xy + 4$

4. Evaluate each of the following polynomials for the indicated value of the variable:

   (i) $x^2 - 25$ for $x = 5$ (ii) $x^2 + 3x - 5$ for $x = -2$

   (iii) $\frac{2}{3}x^3 + \frac{4}{5}x^2 - \frac{7}{5}$ for $x = -1$ (iv) $2x^3 - 3x^2 - 3x + 12$ for $x = -2$

5. Verify that each of $x = 2$ and $x = 3$ is a zero of the polynomial $x^2 - 5x + 6$.

3.7 ADDITION AND SUBTRACTION OF POLYNOMIALS

You are now familiar that polynomials may consist of like and unlike terms. In adding polynomials, we add their like terms together. Similarly, in subtracting a polynomial from another polynomial, we subtract a term from a like term. The question, now, arises 'how do we add or subtract like terms?' Let us take an example.

Suppose we want to add like terms 2x and 3x. The procedure, that we follow in arithmetic, we follow in algebra too. You know that

$$5 \times 6 + 5 \times 7 = 5 \times (6 + 7)$$

$$6 \times 5 + 7 \times 5 = (6 + 7) \times 5$$

Therefore, $2x + 3x = 2 \times x + 3 \times x$

$$= (2 + 3) \times x$$

$$= 5 \times x$$

$$= 5x$$

Similarly, $2xy + 4\,xy = (2 + 4)\,xy = 6xy$

$$3x^2y + 8x^2y = (3 + 8)x^2y = 11x^2y$$

In the same way, since

$$7 \times 5 - 6 \times 5 = (7 - 6) \times 5 = 1 \times 5$$

$$\therefore\ 5y - 2y = (5 - 2) \times y = 3y$$

and $9x^2y^2 - 5x^2y^2 = (9 - 5)x^2y^2 = 4x^2y^2$

In view of the above, we conclude:

1. *The sum of two (or more) like terms is a like term whose numerical coefficient is the sum of the numerical coefficients of the like terms.*
2. *The difference of two like terms is a like term whose numerical coefficient is the difference of the numerical coefficients of the like terms.*

Therefore, to add two or more polynomials, we take the following steps:

Step 1: Group the like terms of the given polynomials together.

Step 2: Add the like terms together to get the sum of the given polynomials.

Example 3.8: Add $-3x + 4$ and $2x^2 - 7x - 2$

Solution:
$$(-3x + 4) + (2x^2 - 7x - 2)$$
$$= 2x^2 + (-3x - 7x) + (4 - 2)$$
$$= 2x^2 + (-3 - 7)x + 2$$
$$= 2x^2 + (-10)x + 2$$
$$= 2x^2 - 10x + 2$$
$$\therefore\ (-3x + 4) + (2x^2 - 7x - 2) = 2x^2 - 10x + 2$$

Polynomials can be added more conveniently if

(i) the given polynomials are so arranged that their like terms are in one column, and

(ii) the coefficients of each column (i.e. of the group of like terms) are added

Thus, Example 3.8 can also be solved as follows:

$$\begin{array}{lll} & -3x & +4 \\ 2x^2 & -7x & -2 \\ \hline 2x^2 + & (-7-3)x & +(4-2) \end{array}$$

$$\therefore\ (-3x + 4) + (2x^2 - 7x - 2) = 2x^2 - 10x + 2$$

Example 3.9: Add $5x + 3y - \frac{3}{4}$ and $-2x + y + \frac{7}{4}$

Solution:
$$\begin{array}{l} 5x + 3y - \frac{3}{4} \\ -2x + y + \frac{7}{4} \\ \hline 3x + 4y + \left(\frac{7}{4} - \frac{3}{4}\right) \\ \hline = 3x + 4y + 1 \end{array}$$

$$\therefore \quad \left(5x + 3y - \frac{3}{4}\right) + \left(-2x + y + \frac{7}{4}\right) = 3x + 4y + 1$$

Example 3.10: Add $\frac{3}{2}x^3 + x^2 + x + 1$ and $x^4 - \frac{x^3}{2} - 3x + 1$

Solution:

$$\frac{3}{2}x^2 + x^2 + x + 1$$

$$+x^4 - \frac{1}{2}x^3 \qquad -3x + 1$$

$$x^4 + \left(\frac{3}{2} - \frac{1}{2}\right)x^3 + x^2 + (1-3)x + (1+1)$$

$$= x^4 + x^3 + x^2 - 2x + 2$$

$$\therefore \left(\frac{3}{2}x^2 + x^2 + x + 1\right) + \left(x^4 - \frac{x^3}{2} - 3x + 1\right) = x^4 + x^3 + x^2 - 2x + 2$$

In order to subtract one polynomial from another polynomial, we go through the following three steps:

Step 1: Arrange the given polynomials in columns so that like terms are in one column.

Step 2: Change the sign (from + to – and from – to +) of each term of the polynomial to be subtracted.

Step 3: Add the like terms of each column separately.

Let us understand the procedure by means of some examples.

Example 3.11: Subtract $-4x^2 + 3x + \frac{2}{3}$ from $9x^2 - 3x - \frac{2}{7}$.

Solution:

$$9x^2 - 3x - \frac{2}{7}$$

$$-4x^2 + 3x + \frac{2}{3}$$

$$+ \qquad - \qquad -$$

$$(9+4)x^2 + (-3-3)x + \left(-\frac{2}{7} - \frac{2}{3}\right)$$

$$= 13x^2 - 6x - \frac{20}{21}$$

$$\therefore \left(9x^2 - 3x - \frac{2}{7}\right) - \left(-4x^2 + 3x + \frac{2}{3}\right) = 13x^2 - 6x - \frac{20}{21}$$

Example 3.12: Subtract $3x - 5x^2 + 7 + 3x^3$ from $2x^2 - 5 + 11x - x^3$.

Solution:

$$\begin{array}{l} -x^3 + 2x^2 + 11x - 5 \\ 3x^3 \;\; - 5x^2 + 3x \;\; + 7 \\ - \quad + \quad - \quad\quad - \\ \hline (-1-3)x^3 + (2+5)x^2 + (11-3)x + (-5-7) \\ \hline = -4x^3 + 7x^2 + 8x - 12 \end{array}$$

$\therefore (2x^2 - 5 + 11x - x^3) - (3x - 5x^2 + 7 + 3x^3) = -4x^3 + 7x^2 + 8x - 12$

Example 3.13: Subtract $12xy - 5y^2 - 9x^2$ from $15xy + 6y^2 + 7x^2$.

Solution:

$$\begin{array}{l} 15xy + 6y^2 + 7x^2 \\ 12xy - 5y^2 - 9x^2 \\ - \quad\; + \quad\; + \\ \hline 3xy + 11y^2 + 16x^2 \\ \hline \end{array}$$

Thus, $(15xy + 6y^2 + 7x^2) - (12xy - 5y^2 - 9x^2) = 3xy + 11y^2 + 16x^2$

We can also directly subtract without arranging expressions in columns as follows:

$(15xy + 6y^2 + 7x^2) - (12xy - 5y^2 - 9x^2)$

$= 15xy + 6y^2 + 7x^2 - 12xy + 5y^2 + 9x^2$

$= 3xy + 11y^2 + 16x^2$

In the same manner, we can add more than two polynomials.

Example 3.14: Add polynomials $3x + 4y - 5x^2$, $5y + 9x$ and $4x - 17y - 5x^2$.

Solution:

$$\begin{array}{l} 3x + 4y - 5x^2 \\ 9x + 5y \\ 4x - 17y - 5x^2 \\ \hline 16x - 8y - 10x^2 \\ \hline \end{array}$$

$\therefore (3x + 4y - 5x^2) + (5y + 9x) + (4x - 17y - 5x^2) = 16x - 8y - 10x^2$

Example 3.15: Subtract $x^2 - x - 1$ from the sum of $3x^2 - 8x + 11$, $-2x^2 + 12x$ and $-4x^2 + 17$.

Solution: Firstly we find the sum of $3x^2 - 8x + 11$, $-2x^2 + 12x$ and $-4x^2 + 17$.

$$\begin{array}{rrr} 3x^2 & -\ 8x & +\ 11 \\ -\ 2x^2 & +\ 12x & \\ -\ 4x^2 & & +\ 17 \\ \hline -\ 3x^2 & +\ 4x & +\ 28 \\ \hline \end{array}$$

Now, we subtract $x^2 - x - 1$ from this sum.

$$\begin{array}{rrr} -\ 3x^2 & +\ 4x & +\ 28 \\ x^2 & -\ \ x & -\ \ 1 \\ - & + & + \\ \hline -\ 4x^2 & +\ 5x & +\ 29 \\ \hline \end{array}$$

Hence, the required result is $-4x^2 + 5x + 29$.

CHECK YOUR PROGRESS 3.3

1. Add the following pairs of polynomials:

   (i) $\frac{2}{3}x^2 + x + 1; \quad \frac{3}{7}x^2 + \frac{1}{4}x + 5$

   (ii) $\frac{7}{5}x^3 - x^2 + 1; \quad 2x^2 + x - 3$

   (iii) $7x^2 - 3x + 4y; \quad 3x^3 + 5x^2 - 4x + \frac{7}{3}y$

   (iv) $2x^3 + 7x^2y - 5xy + 7; \ -2x^2y + 7x^3 - 3xy - 7$

2. Add:

   (i) $x^2 - 3x + 5$, $5 + 7x - 3x^2$ and $x^2 + 7$

   (ii) $\frac{1}{3}x^2 + \frac{7}{8}x - 5$, $\frac{2}{3}x^2 + 5 + \frac{1}{8}x$ and $-x^2 - x$

   (iii) $a^2 - b^2 + ab$, $b^2 - c^2 + bc$ and $c^2 - a^2 + ca$

   (iv) $2a^2 + 3b^2$, $5a^2 - 2b^2 + ab$ and $-6a^2 - 5ab + b^2$

3. Subtract:

   (i) $7x^3 - 3x^2 + 2$ from $x^2 - 5x + 2$

   (ii) $3y - 5y^2 + 7 + 3y^3$ from $2y^2 - 5 + 11y - y^3$

(iii) $2z^3 + 7z - 5z^2 + 2$ from $5z + 7 - 3z^2 + 5z^3$

(iv) $12x^3 - 3x^2 + 11x + 13$ from $5x^3 + 7x^2 + 2x - 4$

4. Subtract $4a - b - ab + 3$ from the sum of $3a - 5b + 3ab$ and $2a + 4b - 5ab$.

3.8 MULTIPLICATION OF POLYNOMIALS

To multiply a monomial by another monomial, we make use of laws of exponents and the rule of signs. For example,

$$3a \times a^2b^2c^2 = (3 \times 1)\, a^{2+1}\, b^2\, c^2 = 3a^3b^2c^2$$

$$-5x \times 2\,xy^3 = (-5 \times 2)\, x^{1+1}\, y^3 = -10\, x^2y^3$$

$$-\frac{1}{2}y^2z \times \left(-\frac{1}{3}\right)yz = \left(-\frac{1}{2}\right)\left(-\frac{1}{3}\right)y^{2+1}z^{1+1} = \frac{1}{6}y^3z^2$$

To multiply a polynomial by a monomial, we multiply each term of the polynomial by the monomial. For example

$$\begin{aligned} x^2y \times (-y^2 + 2xy + 1) &= x^2y \times (-y^2) + (x^2y) \times 2xy + (x^2y) \times 1 \\ &= -x^2y^3 + 2x^3y^2 + x^2y \end{aligned}$$

To multiply a polynomial by another polynomial, we multiply each term of one polynomial by each term of the other polynomial and simplify the result by combining the terms. It is advisable to arrange both the polynomials in increasing or decreasing powers of the variable. For example,

$$\begin{aligned} (2n + 3)(n^2 - 3n + 4) &= 2n \times n^2 + 2n \times (-3n) + 2n \times 4 + 3 \times n^2 + 3 \times (-3n) + 3 \times 4 \\ &= 2n^3 - 6n^2 + 8n + 3n^2 - 9n + 12 \\ &= 2n^3 - 3n^2 - n + 12 \end{aligned}$$

Let us take some more examples.

Example 3.16: Find the product of $(0.2x^2 + 0.7\,x + 3)$ and $(0.5\,x^2 - 3x)$

Solution: $(0.2x^2 + 0.7\,x + 3) \times (0.5\,x^2 - 3x)$

$$= 0.2x^2 \times 0.5\,x^2 + 0.2x^2 \times (-3x) + 0.7\,x \times 0.5\,x^2 + 0.7\,x \times (-3x) + 3 \times 0.5x^2 + 3 \times (-3x)\,)$$

$$= 0.1x^4 - 0.60x^3 + 0.35x^3 - 2.1x^2 + 1.5\,x^2 - 9x$$

$$= 0.1\,x^4 - 0.25\,x^3 - 0.6x^2 - 9x$$

Example 3.17: Multiply $2x - 3 + x^2$ by $1 - x$.

Solution: Arranging polynomials in decreasing powers of x, we get

$$
\begin{aligned}
(x^2 + 2x - 3) \times (-x + 1) &= x^2 \times (-x) + x^2 \times (1) + 2x \times (-x) + 2x \times 1 - 3 \times (-x) \\
&\quad - 3 \times 1 \\
&= -x^3 + x^2 - 2x^2 + 2x + 3x - 3 \\
&= -x^3 - x^2 + 5x - 3
\end{aligned}
$$

Alternative method:

$$
\begin{array}{rrrrl}
 & x^2 & +2x & -3 & \leftarrow \text{one polynomial} \\
 & & -x & +1 & \leftarrow \text{other polynomial} \\
\hline
-x^3 & -2x^2 & +3x & & \\
 & +x^2 & +2x & -3 & \leftarrow \text{Partial products} \\
\hline
-x^3 & -x^2 & +5x & -3 & \leftarrow \text{Product} \\
\hline
\end{array}
$$

3.9 DIVISION OF POLYNOMIALS

To divide a monomial by another monomial, we find the quotient of numerical coefficients and variable(s) separately using laws of exponents and then multiply these quotients. For example,

(i) $$25x^3y^3 \div 5x^2y = \frac{25x^3y^3}{5x^2y} = \frac{25}{5} \times \frac{x^3}{x^2} \times \frac{y^3}{y}$$

$$= 5 \times x^1 \times y^2$$

$$= 5xy^2$$

(ii) $$-12ax^2 \div 4x = -\frac{12ax^2}{4x} = \frac{-12}{4} \times \frac{a}{1} \times \frac{x^2}{x}$$

$$= -3ax$$

To divide a polynomial by a monomial, we divide each term of the polynomial by the monomial. For example,

(i) $$(15x^3 - 3x^2 + 18x) \div 3x = \frac{15x^3}{3x} - \frac{3x^2}{3x} + \frac{18x}{3x}$$

$$= 5x^2 - x + 6$$

(ii) $$(-8x^2 + 10x) \div (-2x) = \frac{-8x^2}{-2x} + \frac{10x}{-2x}$$

$$= \left(\frac{-8}{-2}\right)\left(\frac{x^2}{x}\right) + \frac{10}{(-2)} \times \frac{x}{x}$$

$$= 4x - 5$$

The process of division of a polynomial by another polynomial is done on similar lines as in arithmetic. Try to recall the process when you divided 20 by 3.

$$
\begin{array}{l}
\text{Divisor} \longrightarrow 3\overline{)20} \longleftarrow \text{Dividend} \\
\qquad\qquad\quad 6 \longleftarrow \text{Quotient} \\
\qquad\qquad\quad \underline{18} \\
\qquad\qquad\quad \underline{\ 2\ } \longleftarrow \text{Remainder}
\end{array}
$$

The steps involved in the process of division of a polynomial by another polynomial are explained below with the help of an example.

Let us divide $2x^2 + 5x + 3$ by $2x + 3$.

Step 1: Arrange the terms of both the polynomials in decreasing powers of the variable common to both the polynomials.

$$2x+3\overline{)2x^2+5x+3}$$

Step 2: Divide the first term of the dividend by the first term of the divisor to obtain the first term of the quotient.

$$
\begin{array}{r}
x \qquad\qquad \\
2x+3\overline{)2x^2+5x+3}
\end{array}
$$

Step 3: Multiply all the terms of the divisor by the first term of the quotient and subtract the result from the dividend, to obtain a remainder (as next dividend)

Step 4: Divide the first term of the resulting dividend by the first term of the divisor and write the result as the second term of the quotient.

$$
\begin{array}{r}
x \qquad\qquad \\
2x+3\overline{)2x^2+5x+3} \\
2x^2+3x \qquad \\
\underline{-\quad\ -\qquad} \\
2x+3
\end{array}
$$

Step 5: Multiply all the terms of the divisor by the second term of the quotient and subtract the result from the resulting dividend of Step 4.

Step 6: Repeat the process of Steps 4 and 5, till you get either the remainder zero or a polynomial having the highest exponent of the variable lower than that of the divisor.

$$
\begin{array}{r}
x+1 \qquad\quad \\
2x+3\overline{)2x^2+5x+3} \\
2x^2+3x \qquad \\
\underline{-\quad\ -\qquad} \\
2x+3 \\
2x+3 \\
\underline{-\quad -} \\
\underline{\quad 0 \quad}
\end{array}
$$

In the above example, we got the quotient $x + 1$ and remainder 0.

Let us now consider some more examples.

Example 3.18 : Divide $x^3 - 1$ by $x - 1$.

Solution:

$$
\begin{array}{r}
x^2+x+1 \qquad \\
x-1\overline{)x^3-1} \qquad\quad \\
x^3 \quad -x^2 \\
\underline{-\qquad +\quad} \\
x^2 \quad -1 \\
x^2 \qquad -x \\
\underline{-\qquad\quad +} \\
x \quad -1
\end{array}
$$

$$\begin{array}{r} x \quad -\ 1 \\ - \quad + \\ \hline 0 \\ \hline \end{array}$$

We get quotient $x^2 + x + 1$ and remainder 0.

Example 3.19: Divide $5x - 11 - 12x^2 + 2x^3$ by $2x - 5$.

Solution: Arranging the dividend in decreasing powers of x, we get it as

$2x^3 - 12x^2 + 5x - 11$

So,

$$\begin{array}{r|l} & x^2 - \frac{7}{2}x - \frac{25}{4} \\ \hline 2x-5 & 2x^3 - 12x^2 + 5x - 11 \\ & 2x^3 - 5x^2 \\ & - \qquad + \\ \hline & -7x^2 + 5x - 11 \\ & -7x^2 + \frac{35}{2}x \\ & + \qquad - \\ \hline & -\frac{25}{2}x - 11 \\ & -\frac{25}{2}x + \frac{125}{4} \\ & + \qquad - \\ \hline & -\frac{169}{4} \\ \hline \end{array}$$

We get quotient $x^2 - \frac{7}{2}x - \frac{25}{4}$ and remainder $-\frac{169}{4}$.

CHECK YOUR PROGRESS 3.4

1. Multiply:

(i) $9b^2c^2$ by $3b$ (ii) $5x^3y^5$ by $-2xy$

(iii) $2xy + y^2$ by $-5x$ (iv) $x + 5y$ by $x - 3y$

2. Write the quotient:

(i) $x^5y^3 \div x^2y^2$ (ii) $-28y^7z^2 \div (-4y^3z^2)$

(iii) $(a^4 + a^3b^5) \div a^2$ (iv) $-15b^5c^6 \div 3b^2c^4$

3. Divide and write the quotient and the remainder:

   (i) $x^2 - 1$ by $x + 1$ (ii) $x^2 - x + 1$ by $x + 1$

   (iii) $6x^2 - 5x + 1$ by $2x - 1$ (iv) $2x^3 + 4x^2 + 3x + 1$ by $x + 1$

LET US SUM UP

- A literal number (unknown quantity), which can have various values, is called a variable.
- A constant has a fixed value.
- An algebraic expression is a combination of numbers, variables and arithmetical operations. It has one or more terms joined by the signs + or –.
- Numerical coefficient of a term, say, $2xy$ is 2. Coefficient of x is $2y$ and that of y is $2x$.
- Numerical coefficient of non-negative x is + 1 and that of $-x$ is -1.
- An algebraic expression, in which variable(s) does (do) not occur in the denominator, exponents of variables are whole numbers and numerical coefficients of various terms are real numbers, is called a polynomial.
- The standard form of a polynomial in one variable x is:

  $a_0 + a_1x + a_2x^2 + ... + a_nx^n$ (or written in reverse order) where $a_0, a_1, a_2, ... a_n$ are real numbers and $n, n-1, n-2, ..., 3, 2, 1$ are whole numbers.
- An algebraic expression or a polynomial having one term is called a monomial, that having two terms a bionomial and the one having three terms a trinomial.
- The terms of an algebraic expression or a polynomial having the same variable(s) and same exponent(s) of variable(s) are called like terms. The terms, which are not like, are called unlike terms.
- The sum of the exponents of variables in a term is called the degree of that term.
- The degree of a polynomial is the same as the degree of its term or terms having the highest degree and non-zero numerical coefficient.
- The degree of a non-zero constant polynomial is zero.
- The process of substituting a numerical value for the variable(s) in an algebraic expression (or a polynomial) is called evaluation of the algebraic expression (or polynomial).
- The value(s) of variable(s), for which the value of a polynomial is zero, is (are) called zero(s) of the polynomial.
- The sum of two like terms is a like term whose numerical coefficient is the sum of the numerical coefficients of the two like terms.

- The difference of two like terms is a like term whose numerical coefficient is the difference of the numerical coefficients of the two like terms.
- To multiply or divide a polynomial by a monomial, we multiply or divide each term of the polynomial separately using laws of exponents and the rule of signs.
- To multiply a polynomial by a polynomial, we multiply each term of one polynomial by each term of the other polynomial and simplify the result by combining like terms.
- To divide a polynomial by a polynomial, we usually arrange the terms of both the polynomials in decreasing powers of the variable common to both of them and take steps of division on similar lines as in arithmetic in case of numbers.

TERMINAL EXERCISE

1. Mark a tick (✓) against the correct alternative:

   (i) The coefficient of x^4 in $6x^4y^2$ is

   (A) 6 (B) y^2 (C) $6y^2$ (D) 4

   (ii) Numerical coefficient of the monomial $-x^2y^4$ is

   (A) 2 (B) 6 (C) 1 (D) –1

   (iii) Which of the following algebraic expressions is a polynomial?

   (A) $\frac{1}{\sqrt{2}}x^2 - \sqrt{8} + 3.7x$ (B) $2x + \frac{1}{2x} - 4$

   (C) $(x^2 - 2y^2) \div (x^2 + y^2)$ (D) $6 + \sqrt{x} - x - 15x^2$

   (iv) How many terms does the expression $1 - \sqrt{2}a^2b^3 - (7a)(2b) + \sqrt{3}b^2$ contain?

   (A) 5 (B) 4 (C) 3 (D) 2

   (v) Which of the following expressions is a binomial?

   (A) $2x^2y^2$ (B) $x^2 + y^2 - 2xy$

   (C) $2 + x^2 + y^2 + 2x^2y^2$ (D) $1 - 3xy^3$

   (vi) Which of the following pairs of terms is a pair of like terms?

   (A) 2a, 2b (B) $2xy^3, 2x^3y$

   (C) $3x^2y, \frac{1}{\sqrt{2}}yx^2$ (D) 8, 16 a

   (vii) A zero of the polynomial $x^2 - 2x - 15$ is

   (A) $x = -5$ (B) $x = -3$

(C) $x = 0$ (D) $x = 3$

(viii) The degree of the polynomial $x^3y^4 + 9x^6 - 8y^5 + 17$ is

(A) 7 (B) 17

(C) 5 (D) 6

2. Using variables and operation symbols, express each of the following verbal statements as algebraic statement:

   (i) A number added to itself gives six.

   (ii) Four subtracted from three times a number is eleven.

   (iii) The product of two successive odd numbers is thirty-five.

   (iv) One-third of a number exceeds one-fifth of the number by two.

3. Determine the degree of each of the following polynomials:

   (i) 3^{27} (ii) $x + 7x^2y^2 - 6xy^5 - 18$ (iii) $a^4x + bx^3$ where a and b are constants

   (iv) $c^6 - a^3x^2y^2 - b^2x^3y$ Where a, b and c are constants.

4. Determine whether given value is a zero of the polynomial:

   (i) $x^2 + 3x - 40;$ $x = 8$

   (ii) $x^6 - 1;$ $x = -1$

5. Evaluate each of the following polynomials for the indicated value of the variable:

   (i) $2x - \frac{3}{2}x^2 + \frac{4}{5}x^5 + 7x^3$ at $x = \frac{1}{2}$

   (ii) $\frac{4}{5}y^3 + \frac{1}{5}y^2 - 6y - 65$ at $y = -5$

6. Find the value of $\frac{1}{2}n^2 + \frac{1}{2}n$ for $n = 10$ and verify that the result is equal to the sum of first 10 natural numbers.

7. Add:

   (i) $\frac{7}{3}x^3 + \frac{2}{5}x^2 - 3x + \frac{7}{5}$ and $\frac{2}{3}x^3 + \frac{3}{5}x^2 - 3x + \frac{3}{5}$

   (ii) $x^2 + y^2 + 4xy$ and $2y^2 - 4xy$

   (iii) $x^3 + 6x^2 + 4xy$ and $7x^2 + 8x^3 + y^2 + y^3$

   (iv) $2x^5 + 3x + \frac{2}{3}$ and $-3x^5 + \frac{2}{5}x - 3$

8. Subtract

   (i) $-x^2 + y^2 - xy$ from 0

   (ii) $a + b - c$ from $a - b + c$

   (iii) $x^2 - y^2x + y$ from $y^2x - x^2 - y$

   (iv) $-m^2 + 3mn$ from $3m^2 - 3mn + 8$

9. What should be added to $x^2 + xy + y^2$ to obtain $2x^2 + 3xy$?

10. What should be subtracted from $-13x + 5y - 8$ to obtain $11x - 16y + 7$?

11. The sum of two polynomials is $x^2 - y^2 - 2xy + y - 7$. If one of them is $2x^2 + 3y^2 - 7y + 1$, find the other.

12. If $A = 3x^2 - 7x + 8$, $B = x^2 + 8x - 3$ and $C = -5x^2 - 3x + 2$, find $B + C - A$.

13. Subtract $3x - y - xy$ from the sum of $3x - y + 2xy$ and $-y - xy$. What is the coefficient of x in the result?

14. Multiply

   (i) $a^2 + 5a - 6$ by $2a + 1$ (ii) $4x^2 + 16x + 15$ by $x - 3$

   (iii) $a^2 - 2a + 1$ by $a - 1$ (iv) $a^2 + 2ab + b^2$ by $a - b$

   (v) $x^2 - 1$ by $2x^2 + 1$ (vi) $x^2 - x + 1$ by $x + 1$

   (vii) $x^2 + \frac{2}{3}x + \frac{5}{6}$ by $x - \frac{7}{4}$ (viii) $\frac{2}{3}x^2 + \frac{5}{4}x - 3$ by $3x^2 + 4x + 1$

15. Subtract the product of $(x^2 - xy + y^2)$ and $(x + y)$ from the product of $(x^2 + xy + y^2)$ and $(x - y)$.

16. Divide

   (i) $8x^3 + y^3$ by $2x + y$ (ii) $7x^3 + 18x^2 + 18x - 5$ by $3x + 5$

   (iii) $20x^2 - 15x^3y^6$ by $5x^2$ (iv) $35a^3 - 21a^4b$ by $(-7a^3)$

   (v) $x^3 - 3x^2 + 5x - 8$ by $x - 2$ (vi) $8y^2 + 38y + 35$ by $2y + 7$

   In each case, write the quotient and remainder.

ANSWERS TO CHECK YOUR PROGRESS

3.1

1. (i) y; 1 (ii) x, y; $\frac{2}{3}, \frac{1}{3}, 7$ (iii) x, y; $\frac{4}{5}$

(iv) x, y; $\frac{2}{5}, \frac{1}{2}$ (v) x, y; 2, –8 (vi) x; None

2. (i) 2 (ii) 2y (iii) $2x^2$

3. (i) $x - 3 = 15$ (ii) $x + 5 = 22$

4. (i) 2, abc (ii) a,b,c, 2 (iii) $x^2y, -2xy^2, -\frac{1}{2}$

   (iv) $\frac{1}{8}x^3y^2$

5. (i) $-xy^2, +\frac{1}{3}y^2x$ (ii) –3ab, + ab (iii) No like terms

6. (i) , (ii) and (v) 7. Monomials (ii) and (vi);

   Binomials: (i) and (v); Trinomials : (iii) and (iv)

3.2

1. (i) 7 (ii) 3 (iii) 1 (iv) 0

2. 2.5, 9x, $\frac{2}{9}x^2$, $-25x^3$, $-3x^6$

3. (i) 10 (ii) 4 (iii) 2 (iv) 3

4. (i) 0 (ii) –7 (iii) $-\frac{19}{15}$ (iv) 6

3.3

1. (i) $\frac{23}{11}x^2 + \frac{5}{4}x + 6$ (ii) $\frac{7}{5}x^3 + x^2 + x - 2$

   (iii) $3x^3 + 12x^2 - 7x + \frac{19}{3}y$ (iv) $9x^3 + 5x^2y - 8xy$

2. (i) $-x^2 + 4x + 17$ (ii) 0

   (iii) ab + bc +ca (iv) $a^2 + 2b^2 - 4ab$

3. (i) $-7x^3 + 4x^2 - 5x$ (ii) $-4y^3 + 7y^2 + 8y - 12$

   (iii) $3z^3 + 2z^2 - 2z + 5$ (iv) $-7x^3 + 10x^2 - 9x - 17$

4. a – ab – 3

3.4

1. (i) $27b^3c^2$ (ii) $-10x^4y^6$
   (iii) $-10x^2y-5xy^2$ (iv) $x^2+2xy-15y^2$
2. (i) x^3y (ii) $7y^4$ (iii) a^2+ab^5 (iv) $-5b^3c^2$
3. (i) $x-1; 0$ (ii) $x-2; 3$ (iii) $3x-1; 0$ (iv) $2x^2+2x+1; 0$

ANSWERS TO TERMINAL EXERCISE

1. (i) C (ii) D (iii) A (iv) B (v) D (vi) C (vii) B (viii) A
2. (i) $y+y=6$ (ii) $3y-4=11$ (iii) $z(z+2)=35$ (iv) $\frac{x}{3}-\frac{x}{5}=2$
3. (i) 0 (ii) 6 (iii) 3 (iv) 4
4. (i) No (ii) Yes
5. (i) $\frac{37}{24}$ (ii) 0
6. 55
7. (i) $3x^3+x^2-6x+2$ (ii) x^2+3y^2
   (iii) $9x^3+13x^2+4xy+y^2+y^3$ (iv) $-x^5+\frac{17}{5}x-\frac{7}{3}$
8. (i) x^2-y^2+xy (ii) $2c-2b$
   (iii) $2y^2x-2x^2-2y$ (iv) $4m^2-6mn+8$
9. $x^2+2xy-y^2$
10. $-24x+21y-15$
11. $-x^2-4y^2-2xy+8y-8$
12. $-7x^2+12x-9$
13. $2xy-y; 2y$
14. (i) $2a^3+11a^2-7a-6$ (ii) $4x^3+4x^2-33x-45$
    (iii) a^3-3a^2+3a-1 (iv) $a^3+a^2b-ab^2-b^3$
    (v) $2x^4-x^2-1$ (vi) x^3+1
    (vii) $x^3-\frac{13}{12}x^2-\frac{x}{3}-\frac{35}{24}$ (viii) $2x^4+\frac{77}{12}x^3-\frac{10}{3}x^2-\frac{43}{4}x-3$
15. $-2y^3$
16. (i) $4x^2-2xy+y^2; 0$ (ii) $9x^2-9x+21; -110$
    (iii) $4-3xy^6; 0$ (iv) $-5+3ab; 0$
    (iv) $x^2-x+3; -2$ (v) $4y+5; 0$

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4

SPECIAL PRODUCTS AND FACTORIZATION

In an earlier lesson you have learnt multiplication of algebraic expressions, particularly polynomials. In the study of algebra, we come across certain products which occur very frequently. By becoming familiar with them, a lot of time and labour can be saved as in those products, multiplication is performed without actually writing down all the steps. For example, products, such as 108 × 108, 97 × 97, 104 × 96, 99 × 99 × 99, can be easily calculated if you know the products $(a + b)^2$, $(a - b)^2$, $(a + b)(a - b)$, $(a - b)^3$ respectively. Such products are called **special products**.

Factorization is a process of finding the factors of certain given products such as $a^2 - b^2$, $a^3 + 8b^3$, etc. We will consider factoring only those polynomials in which coefficients are integers.

In this lesson, you will learn about certain special products and factorization of certain polynomials. Besides, you will learn about finding HCF and LCM of polynomials by factorization. In the end you will be made familiar with rational algebraic expressions and to perform fundamental operations on rational expressions.

OBJECTIVES

After studying this lesson, you will be able to

- *write formulae for special products* $(a \pm b)^2$, $(a + b)(a - b)$, $(x + a)(x + b)$, $(a + b)(a^2 - ab + b^2)$, $(a - b)(a^2 + ab + b^2)$, $(a \pm b)^3$ *and* $(ax + b)(cx + d)$;
- *calculate squares and cubes of numbers using formulae;*
- *factorise given polynomials including expressions of the forms* $a^2 - b^2$, $a^3 \pm b^3$;
- *factorise polynomials of the form* $ax^2 + bx + c$ $(a \neq 0)$ *by splitting the middle term;*
- *determine HCF and LCM of polynomials by factorization;*

- *cite examples of rational expressions in one and two variables;*
- *perform four fundamental operations on rational expressions.*

EXPECTED BACKGROUND KNOWLEDGE

- Number system and four fundamental operations
- Laws of exponents
- Algebraic expressions
- Four fundamental operations on polynomials
- HCF and LCM of numbers
- Elementary concepts of geometry and mensuration learnt at primary and upper primary levels.

4.1 SPECIAL PRODUCTS

Here, we consider some speical products which occur very frequently in algebra.

(1) Let us find $(a + b)^2$

$(a + b)^2 = (a + b)(a + b)$

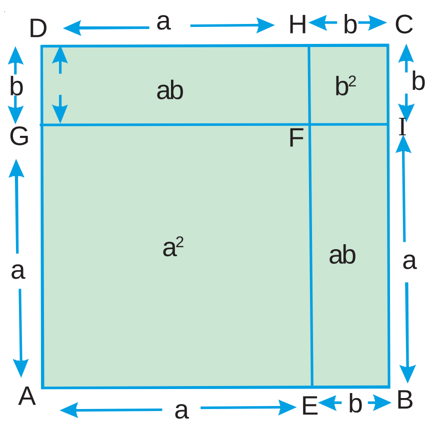
$= a(a + b) + b(a + b)$ [Distributive law]

$= a^2 + ab + ab + b^2$

$= a^2 + 2ab + b^2$

Geometrical verification

Concentrate on the figure, given here, on the right



(i) $(a + b)^2$ = Area of square ABCD

= Area of square AEFG +

area of rectangle EBIF +

area of rectangle DGFH +

area of square CHFI

$= a^2 + ab + ab + b^2$

$= a^2 + 2ab + b^2$

Thus, $\mathbf{(a + b)^2 = a^2 + 2ab + b^2}$

(2) Let us find $(a-b)^2$

$(a-b)^2 = (a-b)(a-b)$ [Distributive law]

$= a(a-b) - b(a-b)$

$= a^2 - ab - ab + b^2$

$= a^2 - 2ab + b^2$

Method 2: Using $(a+b)^2$

We know that $a - b = a + (-b)$

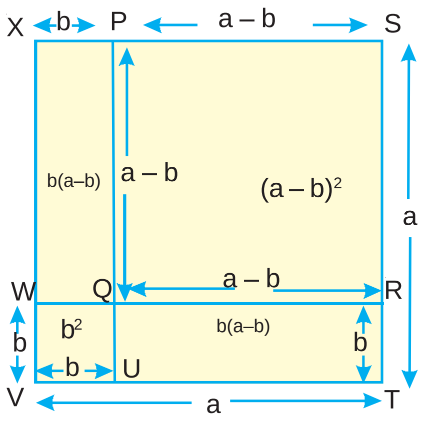
$\therefore (a-b)^2 = [a + (-b)]^2$

$= a^2 + 2(a)(-b) + (-b)^2$

$= a^2 - 2ab + b^2$

Geometrical verification

Concentrate on the figure, given here, on the right



$(a-b)^2$ = Area of square PQRS

= Area of square STVX –

[area of rectangle RTVW +

area of rectangle PUVX –

area of square QUVW]

$= a^2 - (ab + ab - b^2)$

$= a^2 - ab - ab + b^2$

$= a^2 - 2ab + b^2$

Thus, $\mathbf{(a-b)^2 = a^2 - 2ab + b^2}$

Deductions: We have

$(a+b)^2 = a^2 + 2ab + b^2$ .....(1)

$(a-b)^2 = a^2 - 2ab + b^2$ .....(2)

(1) + (2) gives

$\mathbf{(a+b)^2 + (a-b)^2 = 2(a^2 + b^2)}$

(1) – (2) gives

$\mathbf{(a+b)^2 - (a-b)^2 = 4ab}$

(3) Now we find the product (a + b) (a – b)

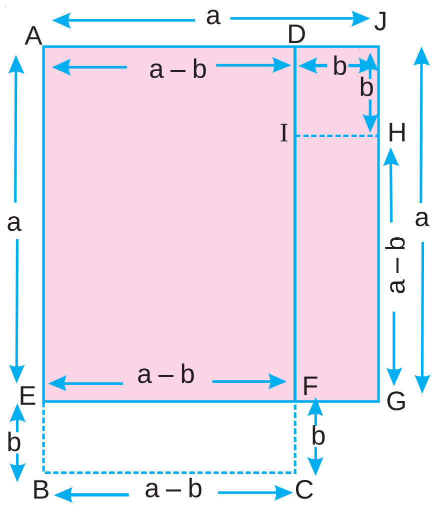
$(a + b)(a - b) = a(a - b) + b(a - b)$ [Distributive law]

$= a^2 - ab + ab - b^2$

$= a^2 - b^2$

Geometrical verification

Observe the figure, given here, on the right



(a + b) (a – b) = Area of Rectangle ABCD

= Area of Rectangle AEFD + area of rectangle EBCF

= Area of Rectangle AEFD + Area of Rectangle FGHI

= [Area of Rectangle AEFD + Area of rectangle FGHI + Area of square DIHJ] – Area of square DIHJ

= Area of square AEGJ – area of square DIHJ

$= a^2 - b^2$

Thus, $\mathbf{(a + b)(a - b) = a^2 - b^2}$

The process of multiplying the sum of two numbers by their difference is very useful in arithmetic. For example,

$64 \times 56 = (60 + 4) \times (60 - 4)$

$= 60^2 - 4^2$

$= 3600 - 16$

$= 3584$

(4) We, now find the product (x + a) (x + b)

$(x + a)(x + b) = x(x + b) + a(x + b)$ [Distributive law]

$= x^2 + bx + ax + ab$

$= x^2 + (a + b)x + ab$

Thus, $\mathbf{(x + a)(x + b) = x^2 + (a + b)x + ab}$

Deductions: (i) $\mathbf{(x - a)(x - b) = x^2 - (a + b)x + ab}$

(ii) $\mathbf{(x - a)(x + b) = x^2 + (b - a)x - ab}$

Students are advised to verify these results.

(5) Let us, now, find the product $(ax + b)(cx + d)$

$$(ax + b)(cx + d) = ax(cx + d) + b(cx + d)$$
$$= acx^2 + adx + bcx + bd$$
$$= acx^2 + (ad + bc)x + bd$$

Thus, $\mathbf{(ax + b)(cx + d) = acx^2 + (ad + bc)x + bd}$

Deductions: (i) $\mathbf{(ax - b)(cx - d) = acx^2 - (ad + bc)x + bd}$

(ii) $\mathbf{(ax - b)(cx + d) = acx^2 - (bc - ad)x - bd}$

Students should verify these results.

Let us, now, consider some examples based on the special products mentioned above.

Example 4.1: Find the following products:

(i) $(2a + 3b)^2$ (ii) $\left(\frac{3}{2}a - 6b\right)^2$

(iii) $(3x + y)(3x - y)$ (iv) $(x + 9)(x + 3)$

(v) $(a + 15)(a - 7)$ (vi) $(5x - 8)(5x - 6)$

(vii) $(7x - 2a)(7x + 3a)$ (viii) $(2x + 5)(3x + 4)$

Solution:

(i) Here, in place of a, we have 2a and in place of b, we have 3b.

$$(2a + 3b)^2 = (2a)^2 + 2(2a)(3b) + (3b)^2$$
$$= 4a^2 + 12ab + 9b^2$$

(ii) Using special product (2), we get

$$\left(\frac{3}{2}a - 6b\right)^2 = \left(\frac{3}{2}a\right)^2 - 2\left(\frac{3}{2}a\right)(6b) + (6b)^2$$

$$= \frac{9}{4}a^2 - 18ab + 36b^2$$

(iii) $(3x + y)(3x - y) = (3x)^2 - y^2$ [using speical product (3)]

$$= 9x^2 - y^2$$

(iv) $(x + 9)(x + 3) = x^2 + (9 + 3)x + 9 \times 3$ [using speical product (4)]

$= x^2 + 12x + 27$

(v) $(a + 15)(a - 7) = a^2 + (15 - 7)a - 15 \times 7$

$= a^2 + 8a - 105$

(vi) $(5x - 8)(5x - 6) = (5x)^2 - (8 + 6)(5x) + 8 \times 6$

$= 25x^2 - 70x + 48$

(vii) $(7x - 2a)(7x + 3a) = (7x)^2 + (3a - 2a)(7x) - (3a)(2a)$

$= 49x^2 + 7ax - 6a^2$

(viii) $(2x + 5)(3x + 4) = (2 \times 3)x^2 + (2 \times 4 + 5 \times 3)x + 5 \times 4$

$= 6x^2 + 23x + 20$

Numerical calculations can be performed more conveniently with the help of special products, often called **algebraic formulae.** Let us consider the following example.

Example 4.2: Using special products, calculate each of the following:

(i) 101×101 (ii) 98×98 (iii) 68×72

(iv) 107×103 (v) 56×48 (vi) 94×99

Solution: (i) $101 \times 101 = 101^2 = (100 + 1)^2$

$= 100^2 + 2 \times 100 \times 1 + 1^2$

$= 10000 + 200 + 1$

$= 10201$

(ii) $98 \times 98 = 98^2 = (100 - 2)^2$

$= 100^2 - 2 \times 100 \times 2 + 2^2$

$= 10000 - 400 + 4$

$= 9604$

(iii) $68 \times 72 = (70 - 2) \times (70 + 2)$

$= 70^2 - 2^2$

$= 4900 - 4$

$= 4896$

(iv) $107 \times 103 = (100 + 7)(100 + 3)$

$= 100^2 + (7 + 3) \times 100 + 7 \times 3$

$= 10000 + 1000 + 21$

$= 11021$

(v) $56 \times 48 = (50+6)(50-2)$

$= 50^2 + (6-2) \times 50 - 6 \times 2$

$= 2500 + 200 - 12$

$= 2688$

(vi) $94 \times 99 = (100-6)(100-1)$

$= 100^2 - (6+1) \times 100 + 6 \times 1$

$= 10000 - 700 + 6$

$= 9306$

CHECK YOUR PROGRESS 4.1

1. Find each of the following products:

(i) $(5x+y)^2$ (ii) $(x-3)^2$ (iii) $(ab+cd)^2$

(iv) $(2x-5y)^2$ (v) $\left(\frac{x}{3}+1\right)^2$ (vi) $\left(\frac{z}{2}-\frac{1}{3}\right)^2$

(vii) $(a^2+5)(a^2-5)$ (viii) $(xy-1)(xy+1)$ (ix) $\left(x+\frac{4}{3}\right)\left(x+\frac{3}{4}\right)$

(x) $\left(\frac{2}{3}x^2-3\right)\left(\frac{2}{3}x^2+\frac{1}{3}\right)$ (xi) $(2x+3y)(3x+2y)$ (xii) $(7x+5y)(3x-y)$

2. Simplify:

(i) $(2x^2+5)^2-(2x^2-5)^2$ (ii) $(a^2+3)^2+(a^2-3)^2$

(iii) $(ax+by)^2+(ax-by)^2$ (iv) $(p^2+8q^2)^2-(p^2-8q^2)^2$

3. Using special products, calculate each of the following:

(i) 102×102 (ii) 108×108 (iii) 69×69

(iv) 998×998 (v) 84×76 (vi) 157×143

(vii) 306×294 (viii) 508×492 (ix) 105×109

(x) 77×73 (xi) 94×95 (xii) 993×996

4.2 SOME OTHER SPECIAL PRODUCTS

(6) Consider the binomial (a + b). Let us find its cube.

$(a + b)^3 = (a + b)(a + b)^2$

$= (a + b)(a^2 + 2ab + b^2)$ [using laws of exponents)

$= a(a^2 + 2ab + b^2) + b(a^2 + 2ab + b^2)$ [Distributive laws)

$= a^3 + 2a^2b + ab^2 + a^2b + 2ab^2 + b^3$

$= a^3 + 3a^2b + 3ab^2 + b^3$

$= a^3 + 3ab(a + b) + b^3$

Thus, $\mathbf{(a + b)^3 = a^3 + 3ab(a + b) + b^3}$

(7) We now find the cube of (a – b).

$(a - b)^3 = (a - b)(a - b)^2$

$= (a - b)(a^2 - 2ab + b^2)$ [using laws of exponents)

$= a(a^2 - 2ab + b^2) - b(a^2 - 2ab + b^2)$ [Distributive laws)

$= a^3 - 2a^2b + ab^2 - a^2b + 2ab^2 - b^3$

$= a^3 - 3a^2b + 3ab^2 - b^3$

$= a^3 - 3ab(a - b) - b^3$

Thus, $\mathbf{(a - b)^3 = a^3 - 3ab(a - b) - b^3}$

Note: You may also get the same result on replacing b by –b in

$(a + b)^3 = a^3 + 3ab(a + b) + b^3$

(8) $(a + b)(a^2 - ab + b^2) = a(a^2 - ab + b^2) + b(a^2 - ab + b^2)$ [Distributive law]

$= a^3 - a^2b + ab^2 + a^2b - ab^2 + b^3$

$= a^3 + b^3$

Thus, $\mathbf{(a + b)(a^2 - ab + b^2) = a^3 + b^3}$

(9) $(a - b)(a^2 + ab + b^2) = a(a^2 + ab + b^2) - b(a^2 + ab + b^2)$ [Distributive law]

$= a^3 + a^2b + ab^2 - a^2b - ab^2 - b^3$

$= a^3 - b^3$

Thus, $\mathbf{(a - b)(a^2 + ab + b^2) = a^3 - b^3}$

Let us, now, consider some examples based on the above mentioned special products:

Example 4.3: Find each of the following products:

(i) $(7x + 9y)^3$ (ii) $(px - yz)^3$ (iii) $(x - 4y^2)^3$

(iv) $(2a^2 + 3b^2)^3$ (v) $\left(\frac{2}{3}a - \frac{5}{3}b\right)^3$ (vi) $\left(1 + \frac{4}{3}c\right)^3$

Solution: (i) $(7x + 9y)^3 = (7x)^3 + 3(7x)(9y)(7x + 9y) + (9y)^3$

$= 343x^3 + 189xy(7x + 9y) + 729y^3$

$= 343x^3 + 1323x^2y + 1701xy^2 + 729y^3$

(ii) $(px - yz)^3 = (px)^3 - 3(px)(yz)(px - yz) - (yz)^3$

$= p^3x^3 - 3pxyz(px - yz) - y^3z^3$

$= p^3x^3 - 3p^2x^2yz + 3pxy^2z^2 - y^3z^3$

(iii) $(x - 4y^2)^3 = x^3 - 3x(4y^2)(x - 4y^2) - (4y^2)^3$

$= x^3 - 12xy^2(x - 4y^2) - 64y^6$

$= x^3 - 12x^2y^2 + 48xy^4 - 64y^6$

(iv) $(2a^2 + 3b^2)^3 = (2a^2)^3 + 3(2a^2)(3b^2)(2a^2 + 3b^2) + (3b^2)^3$

$= 8a^6 + 18a^2b^2(2a^2 + 3b^2) + 27b^6$

$= 8a^6 + 36a^4b^2 + 54a^2b^4 + 27b^6$

(v) $$\left(\frac{2}{3}a - \frac{5}{3}b\right)^3 = \left(\frac{2}{3}a\right)^3 - 3\left(\frac{2}{3}a\right)\left(\frac{5}{3}b\right)\left(\frac{2}{3}a - \frac{5}{3}b\right) - \left(\frac{5}{3}b\right)^3$$

$$= \frac{8}{27}a^3 - \frac{10}{3}ab\left(\frac{2}{3}a - \frac{5}{3}b\right) - \frac{125}{27}b^3$$

$$= \frac{8}{27}a^3 - \frac{20}{9}a^2b + \frac{50}{9}ab^2 - \frac{125}{27}b^3$$

(vi) $$\left(1 + \frac{4}{3}c\right)^3 = (1)^3 + 3(1)\left(\frac{4}{3}c\right)\left(1 + \frac{4}{3}c\right) + \left(\frac{4}{3}c\right)^3$$

$$= 1 + 4c\left(1 + \frac{4}{3}c\right) + \frac{64}{27}c^3$$

$$= 1 + 4c + \frac{16}{3}c^2 + \frac{64}{27}c^3$$

Example 4.4: Using special products, find the cube of each of the following:

(i) 19 (ii) 101 (iii) 54 (iv) 47

Solution: (i) $19^3 = (20 - 1)^3$

$= 20^3 - 3 \times 20 \times 1\ (20 - 1) - 1^3$

$= 8000 - 60\ (20 - 1) - 1$

$= 8000 - 1200 + 60 - 1$

$= 6859$

(ii) $101^3 = (100 + 1)^3$

$= 100^3 + 3 \times 100 \times 1\ (100 + 1) + 1^3$

$= 1000000 + 300 \times 100 + 300 + 1$

$= 1030301$

(iii) $54^3 = (50 + 4)^3$

$= 50^3 + 3 \times 50 \times 4\ (50 + 4) + 4^3$

$= 125000 + 600\ (50 + 4) + 64$

$= 125000 + 30000 + 2400 + 64$

$= 157464$

(iv) $47^3 = (50 - 3)^3$

$= 50^3 - 3 \times 50 \times 3\ (50 - 3) - 3^3$

$= 125000 - 450\ (50 - 3) - 27$

$= 125000 - 22500 + 1350 - 27$

$= 103823$

Example 4.5: Without actual multiplication, find each of the following products:

(i) $(2a + 3b)\ (4a^2 - 6ab + 9b^2)$

(ii) $(3a - 2b)\ (9a^2 + 6ab + 4b^2)$

Solution: (i) $(2a + 3b)\ (4a^2 - 6ab + 9b^2) = (2a + 3b)\ [(2a)^2 - (2a)\ (3b) + (3b)^2]$

$= (2a)^3 + (3b)^3$

$= 8a^3 + 27b^3$

(ii) $(3a - 2b)\ (9a^2 + 6ab + 4b^2) = (3a - 2b)\ [(3a)^2 + (3a)\ (2b) + (2b)^2]$

$= (3a)^3 - (2b)^3$

$= 27a^3 - 8b^3$

Example 4.6: Simplify:

(i) $(3x - 2y)^3 + 3\,(3x - 2y)^2\,(3x + 2y) + 3(3x - 2y)\,(3x + 2y)^2 + (3x + 2y)^3$

(ii) $(2a - b)^3 + 3\,(2a - b)\,(2b - a)\,(a + b) + (2b - a)^3$

Solution: (i) Put $3x - 2y = a$ and $3x + 2y = b$

The given expression becomes

$a^3 + 3a^2b + 3ab^2 + b^3$

$= (a + b)^3$

$= (3x - 2y + 3x + 2y)^3$

$= (6x)^3$

$= 216x^3$

(ii) Put $2a - b = x$ and $2b - a = y$ so that $a + b = x + y$

The given expression becomes

$x^3 + 3xy\,(x + y) + y^3$

$= (x + y)^3$

$= (a + b)^3$

$= a^3 + 3a^2b + 3ab^2 + b^3$

Example 4.7: Simplify:

(i) $\dfrac{857 \times 857 \times 857 - 537 \times 537 \times 537}{857 \times 857 + 857 \times 537 + 537 \times 537}$

(ii) $\dfrac{674 \times 674 \times 674 + 326 \times 326 \times 326}{674 \times 674 - 674 \times 326 + 326 \times 326}$

Solution: The given expression can be written as

$$\frac{857^3 - 537^3}{857^2 + 857 \times 537 + 537^2}$$

Let $857 = a$ and $537 = b$, then the expression becomes

$$\frac{a^3 - b^3}{a^2 + ab + b^2} = \frac{(a - b)(a^2 + ab + b^2)}{a^2 + ab + b^2} = a - b$$

$= 857 - 537$

$= 320$

(ii) The given expression can be written as

$$\frac{674^3 + 326^3}{674^2 - 674 \times 326 + 326^2}$$

$$= \frac{(674+326)(674^2 - 674 \times 326 + 326^2)}{674^2 - 674 \times 326 + 326^2}$$

$= 674 + 326$

$= 1000$

CHECK YOUR PROGRESS 4.2

1. Write the expansion of each of the following:

   (i) $(3x + 4y)^3$ (ii) $(p - qr)^3$ (iii) $\left(a + \frac{b}{3}\right)^3$

   (iv) $\left(\frac{a}{3} - b\right)^3$ (v) $\left(\frac{1}{2}a^2 + \frac{2}{3}b^2\right)^3$ (vi) $\left(\frac{1}{3}a^2x^3 - 2b^3y^2\right)^3$

2. Using special products, find the cube of each of the following:

   (i) 8 (ii) 12 (iii) 18 (iv) 23

   (v) 53 (vi) 48 (vii) 71 (viii) 69

   (ix) 97 (x) 99

3. Without actual multiplication, find each of the following products:

   (i) $(2x + y)(4x^2 - 2xy + y^2)$ (ii) $(x - 2)(x^2 + 2x + 4)$

   (iii) $(1 + x)((1 - x + x^2)$ (iv) $(2y - 3z^2)(4y^2 + 6yz^2 + 9z^4)$

   (v) $(4x + 3y)(16x^2 - 12xy + 9y^2)$ (vi) $\left(3x - \frac{1}{7}y\right)\left(9x^2 + \frac{3}{7}xy + \frac{1}{49}y^2\right)$

4. Find the value of:

   (i) $a^3 + 8b^3$ if $a + 2b = 10$ and $ab = 15$

   [Hint: $(a + 2b)^3 = a^3 + 8b^3 + 6ab(a + 2b) \Rightarrow a^3 + 8b^3 = (a + 2b)^3 - 6ab(a + 2b)$]

   (ii) $x^3 - y^3$ when $x - y = 5$ and $xy = 66$

5. Find the value of $64x^3 - 125z^3$ if

   (i) $4x - 5z = 16$ and $xz = 12$

   (ii) $4x - 5z = \frac{3}{5}$ and $xz = 6$

6. Simplify:

   (i) $(2x + 5)^3 - (2x - 5)^3$

   (ii) $(7x + 5y)^3 - (7x - 5y)^3 - 30y\,(7x + 5y)\,(7x - 5y)^3$

   [Hint put $7x + 5y = a$ and $7x - 5y = b$ so that $a - b = 10y$]

   (iii) $(3x + 2y)\,(9x^2 - 6xy + 4y^2) - (2x + 3y)\,(4x^2 - 6xy + 9y^2)$

   (iv) $(2x - 5)\,(4x^2 + 10x + 25) - (5x + 1)\,(25x^2 - 5x + 1)$

7. Simplify:

   (i) $\dfrac{875 \times 875 \times 875 + 125 \times 125 \times 125}{875 \times 875 - 875 \times 125 + 125 \times 125}$

   (ii) $\dfrac{678 \times 678 \times 678 - 234 \times 234 \times 234}{678 \times 678 + 678 \times 234 + 234 \times 234}$

4.3 FACTORIZATION OF POLYNOMIALS

Recall that from $3 \times 4 = 12$, we say that 3 and 4 are factors of the product 12. Similarly, in algebra, since $(x + y)\,(x - y) = x^2 - y^2$, we say that $(x + y)$ and $(x - y)$ are factors of the product $(x^2 - y^2)$.

Factorization of a polynomial is a process of writing the polynomial as a product of two (or more) polynomials. Each polynomial in the product is called a factor of the given polynomial.

In factorization, we shall restirct ourselves, unless otherwise stated, to finding factors of the polynomials over integers, i.e. polynomials with integral coefficients. In such cases, it is required that the factors, too, be polynomials over integers. Polynomials of the type $2x^2 - y^2$ will not be considered as being factorable into $\left(\sqrt{2}x + y\right)\left(\sqrt{2}x - y\right)$ because these factors are not polynomials over integers.

A polynomial will be said to be completely factored if none of its factors can be further expressed as a product of two polynomials of lower degree and if the integer coefficients have no common factor other than 1 or –1. Thus, complete factorization of $(x^2 - 4x)$ is $x(x-4)$. On the other hand the factorization $(4x^2 - 1)\,(4x^2 + 1)$ of $(16x^4 - 1)$ is not complete since the factor $(4x^2 - 1)$ can be further factorised as $(2x - 1)\,(2x + 1)$. Thus, complete factorization of $(16x^4 - 1)$ is $(2x - 1)\,(2x + 1)\,(4x^2 + 1)$.

In factorization, we shall be making full use of special products learnt earlier in this lesson. Now, in factorization of polynomials we take various cases separately through examples.

(1) Factorization by Distributive Property

Example 4.8: Factorise:

(i) $10a - 25$ (ii) $x^2y^3 + x^3y^2$

(iii) $5ab\,(ax^2 + y^2) - 6mn(ax^2 + y^2)$ (iv) $a(b - c)^2 + b(b - c)$

Solution: (i) $10a - 25 = 5 \times 2a - 5 \times 5$

$= 5\,(2a - 5)$ [Since 5 is common to the two terms]

Thus, 5 and $2a - 5$ are factors of $10a - 25$

(ii) In $x^2y^3 + x^3y^2$, note that x^2y^2 is common (with greatest degree) in both the terms.

$$\therefore\ x^2y^3 + x^3y^2 = x^2y^2 \times y + x^2y^2 \times x$$

$$= x^2y^2\,(y + x)$$

Therefore, $x, x^2, y, y^2, xy, x^2y, xy^2, x^2y^2$ and $y + x$ are factors of $x^2y^3 + x^3y^2$

(iii) Note that $ax^2 + y^2$ is common in both the terms

$$\therefore\ 5ab\,(ax^2 + y^2) - 6mn(ax^2 + y^2) = (ax^2 + y^2)\,(5ab - 6mn)$$

(iv) $a(b - c)^2 + b(b - c) = (b - c) \times [a(b - c)] + (b - c) \times b$

$= (b - c) \times [a(b - c) + b]$

$= (b - c) \times [ab - ac + b]$

(2) Factorization Involving the Difference of Two Squares

You know that $(x + y)\,(x - y) = x^2 - y^2$. Therefore $x + y$ and $x - y$ are factors of $x^2 - y^2$.

Example 4.9: Factorise:

(i) $9x^2 - 16y^2$ (ii) $x^4 - 81y^4$

(iii) $a^4 - (2b - 3c)^2$ (iv) $x^2 - y^2 + 6y - 9$

Solution: (i) $9x^2 - 16y^2 = (3x)^2 - (4y)^2$ which is a difference of two squares.

$= (3x + 4y)\,(3x - 4y)$

(ii) $x^4 - 81y^4 = (x^2)^2 - (9y^2)$

$= (x^2 + 9y^2)\,(x^2 - 9y^2)$

Note that $x^2 - 9y^2 = (x)^2 - (3y)^2$ is again a difference of the two squares.

$$x^4 - 81y^4 = (x^2 + 9y^2)\,[(x)^2 - (3y)^2]$$

$$= (x^2 + 9y^2)\,(x + 3y)\,(x - 3y)$$

(iii) $a^4 - (2b - 3c)^2 = (a^2)^2 - (2b - 3c)^2$

$= [a^2 + (2b - 3c)]\,[a^2 - (2b - 3c)]$

$= (a^2 + 2b - 3c)\,(a^2 - 2b + 3c)$

(iv) $x^2 - y^2 + 6y - 9 \quad = x^2 - (y^2 - 6y + 9)$ [Note this step]

$= (x)^2 - [(y)^2 - 2 \times y \times 3 + (3)^2]$

$= (x)^2 - (y - 3)^2$

$= [x + (y - 3)]\,[x - (y - 3)]$

$= (x + y - 3)\,(x - y + 3)$

(3) Factorization of a Perfect Square Trinomial

Example 4.10 : Factorise

(i) $9x^2 + 24xy + 16y^2$ (ii) $x^6 - 8x^3 + 16$

Solution: (i) $9x^2 + 24xy + 16y^2 \quad = (3x)^2 + 2\,(3x)\,(4y) + (4y)^2$

$= (3x + 4y)^2$

$= (3x + 4y)\,(3x + 4y)$

Thus, the two factors of the given polynomial are identical, each being $(3x + 4y)$.

(ii) $x^6 - 8x^3 + 16 \quad = (x^3)^2 - 2(x^3)\,(4) + (4)^2$

$= (x^3 - 4)^2$

$= (x^3 - 4)\,(x^3 - 4)$

Again, the two factors of the given polynomial are identical, each being $(x^3 - 4)$.

(4) Factorization of a Polynomial Reducible to the Difference of Two Squares

Example 4.11: Factorise

(i) $x^4 + 4y^4$ (ii) $x^4 + x^2 + 1$

Solution: (i) $x^4 + 4y^4 \quad = (x^2)^2 + (2y^2)^2$

$= (x^2)^2 + (2y^2)^2 + 2\,(x^2)\,(2y^2) - 2\,(x^2)\,(2y^2)$

[Adding and subtracting $2\,(x^2)\,(2y^2)$]

$= (x^2 + 2y^2)^2 - (2xy)^2$

$= (x^2 + 2y^2 + 2xy)\,(x^2 + 2y^2 - 2xy)$

(ii) $x^4 + x^2 + 1 = (x^2)^2 + (1)^2 + 2x^2 - x^2$

[Adding and subtracting x^2]

$= (x^2 + 1)^2 - (x)^2$

$= (x^2 + x + 1)(x^2 - x + 1)$

CHECK YOUR PROGRESS 4.3

Factorise:

1. $10xy - 15xz$
2. $abc^2 - ab^2c$
3. $6p^2 - 15pq + 27p$
4. $a^2(b - c) + b(c - b)$
5. $2a(4x - y)^3 - b(4x - y)^2$
6. $x(x + y)^3 - 3xy(x + y)$
7. $100 - 25p^2$
8. $1 - 256y^8$
9. $(2x + 1)^2 - 9x^2$
10. $(a^2 + bc)^2 - a^2(b + c)^2$
11. $25x^2 - 10x + 1 - 36y^2$
12. $49x^2 - 1 - 14xy + y^2$
13. $m^2 + 14m + 49$
14. $4x^2 - 4x + 1$
15. $36a^2 + 25 + 60a$
16. $x^6 - 8x^3 + 16$
17. $a^8 - 47a^4 + 1$
18. $4a^4 + 81b^4$
19. $x^4 + 4$
20. $9a^4 - a^2 + 16$
21. Find the value of n if

   (i) $6n = 23 \times 23 - 17 \times 17$ (ii) $536 \times 536 - 36 \times 36 = 5n$

(5) Factorization of Perfect Cube Polynomials

Example 4.12: Factorise:

(i) $x^3 + 6x^2y + 12xy^2 + 8y^3$ (ii) $x^6 - 3x^4y^2 + 3x^2y^4 - y^6$

Solution: (i) $x^3 + 6x^2y + 12xy^2 + 8y^3$

$= (x)^3 + 3x^2(2y) + 3x(2y)^2 + (2y)^3$

$= (x + 2y)^3$

Thus, the three factors of the given polynomial are identical, each being $x + 2y$.

(ii) Given polynomial is equal to

$(x^2)^3 - 3x^2y^2(x^2 - y^2) - (y^2)^3$

$= (x^2 - y^2)^3$

$= [(x + y)(x - y)]^3$ [Since $x^2 - y^2 = (x + y)(x - y)$]

$= (x + y)^3 (x - y)^3$

(6) Factorization of Polynomials Involving Sum or Difference of Two Cubes

In special products you have learnt that

$$(x + y)(x^2 - xy + y^2) = x^3 + y^3$$

and $$(x - y)(x^2 + xy + y^2) = x^3 - y^3$$

Therefore, the factors of $x^3 + y^3$ are $x + y$ and $x^2 - xy + y^2$ and those of $x^3 - y^3$ are $x - y$ and $x^2 + xy + y^2$

Now, consider the following example:

Example 4.13: Factorise

(i) $64a^3 + 27b^3$ (ii) $8x^3 - 125y^3$

(iii) $8(x + 2y)^3 - 343$ (iv) $a^4 - a^{13}$

Solution: (i) $64a^3 + 27b^3 = (4a)^3 + (3b)^3$

$= (4a + 3b)[(4a)^2 - (4a)(3b) + (3b)^2]$

$= (4a + 3b)(16a^2 - 12ab + 9b^2)$

(ii) $8x^3 - 125y^3 = (2x)^3 - (5y)^3$

$= (2x - 5y)[(2x)^2 + (2x)(5y) + (5y)^2]$

$= (2x - 5y)(4x^2 + 10xy + 25y^2)$

(iii) $8(x + 2y)^3 - 343 = [2(x + 2y)]^3 - (7)^3$

$= [2(x + 2y) - 7][2^2(x + 2y)^2 + 2(x + 2y)(7) + 7^2]$

$= (2x + 4y - 7)(4x^2 + 16xy + 16y^2 + 14x + 28y + 49)$

(iv) $a^4 - a^{13} = a^4(1 - a^9)$ [Since a^4 is common to the two terms]

$= a^4[(1)^3 - (a^3)^3]$

$= a^4(1 - a^3)(1 + a^3 + a^6)$

$= a^4(1 - a)(1 + a + a^2)(1 + a^3 + a^6)$

[Since $1 - a^3 = (1 - a)(1 + a + a^2)$]

CHECK YOUR PROGRESS 4.4

Factorise:

1. $a^3 + 216b^3$
2. $a^3 - 343$
3. $x^3 + 12x^2y + 48xy^2 + 64y^3$
4. $8x^3 - 36x^2y + 54xy^2 - 27y^3$

5. $8x^3 - 125y^3 - 60x^2y + 150xy^2$
6. $64k^3 - 144k^2 + 108k - 27$
7. $729\,x^6 - 8$
8. $x^2 + x^2\,y^6$
9. $16a^7 - 54ab^6$
10. $27b^3 - a^3 - 3a^2 - 3a - 1$
11. $(2a - 3b)^3 + 64c^3$
12. $64x^3 - (2y - 1)^3$

(7) Factorising Trinomials by Splitting the Middle Term

You have learnt that

$$(x + a)(x + b) \quad = x^2 + (a + b)x + ab \quad = 1.x^2 + (a + b)x + ab$$

and $(ax + b)(cx + d) \quad = acx^2 + (ad + bc)x + bd$

In general, the expressions given here on the right are of the form $Ax^2 + Bx + C$ which can be factorised by multiplying the coefficient of x^2 in the first term with the last term and finding two such factors of this product that their sum is equal to the coefficient of x in the second (middle) term. In other words, we are to determine two such factors of AC so that their sum is equal to B. The example, given below, will clarify the process further.

Example 4.14: Factorise:

(i) $x^2 + 3x + 2$ (ii) $x^2 - 10xy + 24y^2$

(iii) $5x^2 + 13x - 6$ (iv) $3x^2 - x - 2$

Solution: (i) Here, A = 1, B = 3 and C= 2; so AC = $1 \times 2 = 2$

Therefore we are to determine two factors of 2 whose sum is 3

Obviously, $1 + 2 = 3$

(i.e. two factors of AC i.e. 2 are 1 and 2)

$\therefore$ We write the polynomial as

$$x^2 + (1 + 2)\,x + 2$$
$$= x^2 + x + 2x + 2$$
$$= x(x + 1) + 2(x + 1)$$
$$= (x + 1)(x + 2)$$

(ii) Here, AC = $24y^2$ and B = $-10y$

Two factors of $24y^2$ whose sum is $-10y$ are $-4y$ and $-6y$

$\therefore$ We write the given polynomial as

$$x^2 - 4xy - 6xy + 24\,y^2$$
$$= x(x - 4y) - 6y(x - 4y)$$
$$= (x - 4y)(x - 6y)$$

(iii) Here, AC = $5 \times (-6) = -30$ and B = 13

Two factors of -30 whose sum is 13 are 15 and -2

$\therefore$ We write the given polynomial as

$5x^2 + 15x - 2x - 6$

$= 5x(x+3) - 2(x+3)$

$= (x+3)(5x-2)$

(iv) Here, AC = $3 \times (-2) = -6$ and B = -1

Two factors of -6 whose sum is (-1) are (-3) and 2.

$\therefore$ We write the given polynomial as

$3x^2 - 3x + 2x - 2$

$= 3x(x-1) + 2(x-1)$

$= (x-1)(3x+2)$

CHECK YOUR PROGRESS 4.5

Factorise:

1. $x^2 + 11x + 24$
2. $x^2 - 15xy + 54y^2$
3. $2x^2 + 5x - 3$
4. $6x^2 - 10xy - 4y^2$
5. $2x^4 - x^2 - 1$
6. $x^2 + 13xy - 30y^2$
7. $2x^2 + 11x + 14$
8. $10y^2 + 11y - 6$
9. $2x^2 - x - 1$
10. $(m-1)(1-m) + m + 109$
11. $(2a-b)^2 - (2a-b) - 30$

    Hint put $2a - b = x$
12. $(2x+3y)^2 - 2(2x+3y)(3x-2y) - 3(3x-2y)^2$

    Hint: Put $2x + 3y = a$ and $3x - 2y = b$

4.4 HCF AND LCM OF POLYNOMIALS

(1) HCF of Polynomials

You are already familiar with the term HCF (Highest Common Factor) of natural numbers in arithmetic. It is the largest number which is a factor of each of the given numbers. For instance, the HCF of 8 and 12 is 4 since the common factors of 8 and 12 are 1, 2 and 4 and 4 is the largest i.e. highest among them.

On similar lines in algebra, *the Highest Common Factor (HCF) of two or more given* *polynomials is the product of the polynomial(s) of highest degree and greatest numerical coefficient each of which is a factor of each of the given polynomials.*

For example, the HCF of $4(x + 1)^2$ and $6(x + 1)^3$ is $2(x + 1)^2$.

The HCF of monomials is found by multiplying the HCF of numerical coefficients of each of the monomials and the variable(s) with highest power(s) common to all the monomials. For example, the HCF of monomials $12x^2y^3$, $18xy^4$ and $24x^3y^5$ is $6xy^3$ since HCF of 12, 18 and 24 is 6; and the highest powers of variable factors common to the polynomials are x and y^3.

Let us now consider some examples.

Example 4.15: Find the HCF of

(i) $4x^2y$ and x^3y^2 (ii) $(x - 2)^3 (2x - 3)$ and $(x - 2)^2 (2x - 3)^3$

Solution: (i) HCF of numerical coefficients 4 and 1 is 1.

Since x occurs as a factor at least twice and y at least once in the given polynomials, therefore, their HCF is

$$1 \times x^2 \times y \text{ i.e. } x^2y$$

(ii) HCF of numerical coefficients 1 and 1 is 1.

In the given polynomials, $(x - 2)$ occurs as a factor at least twice and $(2x - 3)$ at least once. So the HCF of the given polynomials is

$$1 \times (x - 2)^2 \times (2x - 3) \text{ i.e. } (x - 2)^2 (2x - 3)$$

In view of Example 4.15 (ii), we can say that to determine the HCF of polynomials, which can be easily factorised, we express each of the polynomials as the product of the factors. Then the HCF of the given polynomials is the product of the HCF of numerical coefficients of each of the polynomials and factor (s) with highest power(s) common to all the polynomials. For further clarification, concentrate on the Example 4.16 given below.

Example 4.16: Find the HCF of

(i) $x^2 - 4$ and $x^2 + 4x + 4$

(ii) $4x^4 - 16x^3 + 12x^2$ and $6x^3 + 6x^2 - 72x$

Solution: (i) $x^2 - 4 = (x + 2)(x - 2)$

$x^2 + 4x + 4 = (x + 2)^2$

HCF of numerical coefficients = 1

HCF of other factors = $(x + 2)^1 = x + 2$

Hence, the required HCF = $x + 2$

(ii) $4x^4 - 16x^3 + 12x^2 = 4x^2(x^2 - 4x + 3)$

$= 4x^2(x - 1)(x - 3)$

$$6x^3 + 6x^2 - 72x \qquad = 6x(x^2 + x - 12)$$
$$= 6x(x + 4)(x - 3)$$

Required HCF $= 2x(x - 3)$ [Since HCF of numerical coefficient is 2)

$= 2x^2 - 6x$

(2) LCM of Polynomials

Like HCF, you are also familiar with the LCM (Lowest Common Multiple or Least Common Multiple) of natural numbers in arithmetic. It is the smallest number which is a multiple of each of the given numbers. For instance, the LCM of 8 and 12 is 24 since 24 is the smallest among common multiples of 8 and 12 as given below:

Multiples of 8: 8, 16, $\underline{24}$, 32, 40, $\underline{48}$, 56, 64, $\underline{72}$, 80, ...

Multiples of 12: 12, $\underline{24}$, 36, $\underline{48}$, 60, $\underline{72}$, 84, 96, ...

Common multiple of 8 and 12: 24, 48, 72, ...

On similar lines in Algebra, *the Lowest Common Multiple (LCM) of two or more polynomials is the product of the polynomial(s) of the lowest degree and the smallest numerical coefficient which are multiples of the corresponding elements of each of the given polynomials.*

For example, the LCM of $4(x + 1)^2$ and $6(x + 1)^3$ is $12(x + 1)^3$.

The LCM of monomials is found by multiplying the LCM of numerical coefficients of each of the monomials and all variable factors with highest powers. For example, the LCM of $12x^2y^2z$ and $18x^2yz$ is $36x^2y^2z$ since the LCM of 12 and 18 is 36 and highest powers variable factors x, y and z are x^2, y^2 and z respectively.

Let us, now, consider some examples to illustrate.

Example 4.17: Find the LCM of

(i) $4x^2y$ and x^3y^2 (ii) $(x - 2)^3(2x - 3)$ and $(x - 2)^2(2x - 3)^3$

Solution: (i) LCM of numerical coefficient 4 and 1 is 4.

Since highest power of x is x^3 and that of y is y^2,

the required LCM is $4x^3y^2$

(ii) Obviously LCM of numerical coefficients 1 and 1 is 1.

In the given polynomials, highest power of the factor $(x - 2)$ is $(x - 2)^3$

and that of $(2x - 3)$ is $(2x - 3)^3$.

LCM of the given polynomials $= 1 \times (x - 2)^3 \times (2x - 3)^3$

$= (x - 2)^3(2x - 3)^3$

In view of Example 4.17 (ii), we can say that to determine the LCM of polynomials, which can be easily factorised, we express each of the polynomials as the product of factors. Then, the LCM of the given polynomials is the product of the LCM of the numerical coefficients and all other factors with their highest powers which occur in factorization of any of the polynoials. For further clarification, we take Example 4.18 given below.

Example 4.18: Find the LCM of

(i) $(x-2)(x^2-3x+2)$ and x^2-5x+6

(ii) $8(x^3-27)$ and $12(x^5+27x^2)$

Solution: (i) $(x-2)(x^2-3x+2) = (x-2)(x-2)(x-1)$

$= (x-2)^2(x-1)$

Also $x^2-5x+6 = (x-2)(x-3)$

LCM of numerical coefficients = 1

LCM of other factors = $(x-2)^2(x-1)(x-3)$

Hence, the LCM of given polynomials = $(x-1)(x-2)^2(x-3)$

(ii) $8(x^3-27) = 8(x-3)(x^2+3x+9)$

$12(x^5+27x^2) = 12x^2(x^3+27)$

$= 12x^2(x+3)(x^2-3x+9)$

LCM of numerical coefficient 8 and 12 = 24

LCM of other factors = $x^2(x-3)(x+3)(x^2+3x+9)(x^2-3x+9)$

Hence, required LCM = $24x^2(x-3)(x+3)(x^2+3x+9)(x^2-3x+9)$

CHECK YOUR PROGRESS 4.6

1. Find the HCF of the following polynomials:

(i) $27x^4y^2$ and $3xy^3$ (ii) $48y^7x^9$ and $12y^3x^5$

(iii) $(x+1)^3$ and $(x+1)^2(x-1)$ (iv) x^2+4x+4 and $x+2$

(v) $18(x+2)^3$ and $24(x^3+8)$ (vi) $(x+1)^2(x+5)^3$ and $x^2+10x+25$

(vii) $(2x-5)^2(x+4)^3$ and $(2x-5)^3(x-4)$ (viii) x^2-1 and x^4-1

(ix) x^3-y^3 and x^2-y^2 (x) $6(x^2-3x+2)$ and $18(x^2-4x+3)$

2. Find the LCM of the following polynomials:

(i) $25x^3y^2$ and $15xy$ (ii) $30\,xy^2$ and $48\,x^3y^4$

(iii) $(x+1)^3$ and $(x+1)^2(x-1)$ (iv) x^2+4x+4 and $x+2$

(v) $18(x+2)^3$ and $24(x^3+8)$ (vi) $(x+1)^2(x+5)^3$ and $x^2+10x+25$

(vii) $(2x-5)^2(x+4)^2$ and $(2x-5)^3(x-4)$ (viii) x^2-1 and x^4-1

(ix) x^3-y^3 and x^2-y^2 (x) $6(x^2-3x+2)$ and $18(x^2-4x+3)$

4.5 RATIONAL EXPRESSIONS

You are already familiar with integers and rational numbers. Just as a number, which can be expressed in the form $\frac{p}{q}$ where p and q ($\neq 0$) are integers, is called a rational number, an algebraic expression, which can be expressed in the from $\frac{P}{Q}$, where P and Q (non-zero polynomials) are polynomials, is called a **rational expression.** Thus, each of the expressions

$$\frac{x+1}{x-1}, \frac{x^2-3x+5}{x^2-5}, \frac{\frac{1}{2}a^2+b^2-\frac{5}{6}}{a+b}, \frac{x^2+\sqrt{2}y^2}{\sqrt{3x-y}}$$

is a rational expression in one or two variables.

Notes:

(1) The polynomial '$x^2 + 1$' is a rational expresion since it can be written as $\frac{x^2+1}{1}$ and you have learnt that the constant 1 in the denominator is a polynomial of degree zero.

(2) The polynomial 7 is a rational expresion since it can be written as $\frac{7}{1}$ where both 7 and 1 are polynomials of degree zero.

(3) Obvioulsy a rational expression need not be a polynomial. For example rational expression $\frac{1}{x}\left(= x^{-1}\right)$ is not a polynomial. On the contrary every polynomial is also a rational expression.

None of the expressions $\frac{\sqrt{x}+2}{1-x}$, $x^2+2\sqrt{x}+3$, $\frac{a^{\frac{2}{3}}-\frac{1}{b}}{a^2+ab+b^2}$ is a rational expression.

CHECK YOUR PROGRESS 4.7

1. Which of the following algebraic expressions are rational expressions?

(i) $\frac{2x-3}{4x-1}$ (ii) $\frac{8}{x^2+y^2}$

(iii) $\dfrac{2\sqrt{3}x^2+\sqrt{5}}{\sqrt{7}}$ (iv) $\dfrac{2x^2-\sqrt{x}+3}{6x}$

(v) $200+\sqrt{11}$ (vi) $\left(a+\dfrac{1}{b}\right)\div b^{\frac{1}{3}}$

(vii) $y^3+3yz\,(y+z)+z^3$ (vii) $5\div(a+3b)$

2. For each of the following, cite two examples:
   (i) A rational expression is one variable
   (ii) A rational expression is two variables
   (iii) A rational expression whose numerator is a binomial and whose denominator is trinomial
   (iv) A rational expression whose numerator is a constant and whose denominator is a quadratic polynomial
   (v) A rational expression in two variables whose numerator is a polynomial of degree 3 and whose denominator is a polynomial of degree 5
   (vi) An algebraic expression which is not a rational expression

4.6 OPERATIONS ON RATIONAL EXPRESSIONS

Four fundamental operations on rational expressions are performed in exactly the same way as in case of rational numbers.

(1) Addition and Subtraction of Rational Expressions

For observing the analogy between addition of rational numbers and that of rational expressions, we take the following example. Note that the analogy will be true for subtraction, multiplication and division of rational expressions also.

Example 4.19: Find the sum:

(i) $\dfrac{5}{6}+\dfrac{3}{8}$ (ii) $\dfrac{2x+1}{x-1}+\dfrac{x+2}{x+1}$

Solution: (i) $\dfrac{5}{6}+\dfrac{3}{8} = \dfrac{5\times4+3\times3}{24}$ ← LCM of 6 and 8.

$$= \frac{20+9}{24}$$

$$= \frac{29}{24}$$

(ii) $\frac{2x+1}{x-1}+\frac{x+2}{x+1}$ $= \frac{(2x+1)(x+1)+(x+2)(x-1)}{(x-1)(x+1)}$ ← LCM of (x – 1) and (x + 1)

$= \frac{2x^2+3x+1+x^2+x-2}{x^2-1}$

$= \frac{3x^2+4x-1}{x^2-1}$

Example 4.20: Subtract $\frac{x-1}{x+1}$ from $\frac{3x-2}{3x+1}$

Solution: $\frac{3x-2}{3x+1}-\frac{x-1}{x+1}$ $= \frac{(x+1)(3x-2)-(x-1)(3x+1)}{(3x+1)(x+1)}$

$= \frac{3x^2+x-2-(3x^2-2x-1)}{3x^2+4x+1}$

$= \frac{3x-1}{3x^2+4x+1}$

Note: Observe that the sum and difference of two rational expressions are also rational expressions.

Since the sum and difference of two rational expressions are rational expressions, $x+\frac{1}{x}(x\neq 0)$ and $x-\frac{1}{x}(x\neq 0)$ are both rational expressions as x and $\frac{1}{x}$ are both rational expressions. Similarly, each of $x^2+\frac{1}{x^2}, x^3+\frac{1}{x^3}, x^2-\frac{1}{x^2}, x^3-\frac{1}{x^3}$, etc. is a rational expression. These expresions create interest as for given value of $x+\frac{1}{x}$ or $x-\frac{1}{x}$, we can determine values of $x^2+\frac{1}{x^2}, x^2-\frac{1}{x^2}, x^3+\frac{1}{x^3}, x^3-\frac{1}{x^3}$ etc. and in some case vice versa also. Let us concentrate on the following example.

Example 4.21: Find the value of

(i) $x^2+\frac{1}{x^2}$ if $x-\frac{1}{x}=1$

(ii) $x^4+\frac{1}{x^4}$ if $x+\frac{1}{x}=4$

(iii) $x-\frac{1}{x}$ if $x^4+\frac{1}{x^4}=119$

(iv) $x^3+\frac{1}{x^3}$ if $x+\frac{1}{x}=3$

(v) $x^3 - \frac{1}{x^3}$ if $x - \frac{1}{x} = 5$

Solution: (i) We have $x - \frac{1}{x} = 1$

$$\therefore \left(x - \frac{1}{x}\right)^2 = (1)^2$$

$$\Rightarrow x^2 + \frac{1}{x^2} - 2 \times x \times \frac{1}{x} = 1$$

$$\Rightarrow x^2 + \frac{1}{x^2} - 2 = 1$$

Hence, $x^2 + \frac{1}{x^2} = 3$

(ii) $x + \frac{1}{x} = 4$

$$\Rightarrow \left(x + \frac{1}{x}\right)^2 = (4)^2$$

$$\Rightarrow x^2 + \frac{1}{x^2} + 2 = 16$$

$$\Rightarrow x^2 + \frac{1}{x^2} = 14$$

$$\Rightarrow \left(x^2 + \frac{1}{x^2}\right)^2 = (14)^2$$

$$\Rightarrow x^4 + \frac{1}{x^4} + 2 = 196$$

So, $x^4 + \frac{1}{x^4} = 194$

(iii) We have $x^4 + \frac{1}{x^4} = 119$

$$\Rightarrow (x^2)^2 + \left(\frac{1}{x^2}\right)^2 + 2 = 119 + 2 = 121$$

$$\Rightarrow \left(x^2+\frac{1}{x^2}\right)^2=(11)^2$$

$$\Rightarrow x^2+\frac{1}{x^2}=11 \qquad \left[\text{since both } x^2 \text{ and } \frac{1}{x^2} \text{ are positive}\right]$$

$$\Rightarrow x^2+\frac{1}{x^2}-2=9$$

$$\Rightarrow \left(x-\frac{1}{x}\right)^2=(3)^2$$

$$\therefore x-\frac{1}{x}=\pm 3$$

(iv) We have $x+\frac{1}{x}=3$

$$\therefore \left(x+\frac{1}{x}\right)^3=(3)^3$$

$$\Rightarrow x^3+\frac{1}{x^3}+3\times x\times\frac{1}{x}\left(x+\frac{1}{x}\right)=27$$

$$\Rightarrow x^3+\frac{1}{x^3}+3(3)=27$$

$$\therefore x^3+\frac{1}{x^3}=18$$

(v) We have $x-\frac{1}{x}=5$

$$\therefore \left(x-\frac{1}{x}\right)^3=(5)^3$$

$$\Rightarrow x^3-\frac{1}{x^3}-3\times x\times\frac{1}{x}\left(x-\frac{1}{x}\right)=125$$

$$\Rightarrow x^3-\frac{1}{x^3}-3(5)=125$$

$$\therefore x^3-\frac{1}{x^3}=140$$

CHECK YOUR PROGRESS 4.8

1. Find the sum of rational expressions:

(i) $\frac{x^2+1}{x-2}$ and $\frac{x^2-1}{x-2}$ (ii) $\frac{x+2}{x+3}$ and $\frac{x-1}{x-2}$

(iii) $\frac{x+1}{(x-1)^2}$ and $\frac{1}{x+1}$ (iv) $\frac{3x+2}{x^2-16}$ and $\frac{x-5}{(x+4)^2}$

(v) $\frac{x-2}{x+3}$ and $\frac{x+2}{x+3}$ (vi) $\frac{x+2}{x-2}$ and $\frac{x-2}{x+2}$

(vii) $\frac{x+1}{x+2}$ and $\frac{x^2-1}{x^2+1}$ (vii) $\frac{3\sqrt{2}x+1}{3x^2}$ and $\frac{-2\sqrt{2}x+1}{2x^2}$

2. Subtract

(i) $\frac{x-1}{x-2}$ from $\frac{x+4}{x+2}$ (ii) $\frac{2x-1}{2x+1}$ from $\frac{2x+1}{2x-1}$

(iii) $\frac{1}{x}$ from x (iv) $\frac{2}{x}$ from $\frac{x+1}{x^2-1}$

(v) $\frac{x^2+1}{x-4}$ from $\frac{2x^2+3}{x-4}$ (vi) $\frac{1}{x^2+2}$ from $\frac{2x^3+x^2+3}{(x^2+2)^2}$

(vii) $\frac{x+2}{2(x^2-9)}$ from $\frac{x-2}{(x+3)^2}$ (vii) $\frac{x+1}{x-1}$ from $\frac{4x}{x^2-1}$

3. Find the value of

(i) $a^2+\frac{1}{a^2}$ when $a+\frac{1}{a}=2$ (ii) $a^2+\frac{1}{a^2}$ when $a-\frac{1}{a}=2$

(iii) $a^3+\frac{1}{a^3}$ when $a+\frac{1}{a}=2$ (iv) $a^3+\frac{1}{a^3}$ when $a+\frac{1}{a}=5$

(v) $a^3-\frac{1}{a^3}$ when $a-\frac{1}{a}=\sqrt{5}$ (vi) $8a^3+\frac{1}{27a^3}$ when $2a+\frac{1}{3a}=5$

(vii) $a^3+\frac{1}{a^3}$ when $a+\frac{1}{a}=\sqrt{3}$ (viii) $a^3+\frac{1}{a^3}$ when $a^2+\frac{1}{a^2}=7, a>0$

(ix) $a-\frac{1}{a}$ when $a^4+\frac{1}{a^4}=727$ (x) $a^3-\frac{1}{a^3}$ when $a^4+\frac{1}{a^4}=34, a>0$

(2) Multiplication and Division of Rational Expressions

You know that the product of two rational numbers, say, $\frac{2}{3}$ and $\frac{5}{7}$ is given as $\frac{2}{3}\times\frac{5}{7}=\frac{2\times5}{3\times7}=\frac{10}{21}$. Similarly, the product of two rational expressions, say, $\frac{P}{Q}$ and $\frac{R}{S}$ where P, Q, R, S (Q, S $\neq 0$) are polynomials is given by $\frac{P}{Q}\times\frac{R}{S}=\frac{PR}{QS}$. You may observe that the product of two rational expressions is again a rational expression.

Example 4.22: Find the product:

(i) $\frac{5x+3}{5x-1}\times\frac{2x-1}{x+1}$ (ii) $\frac{2x+1}{x-1}\times\frac{x-1}{x+3}$

(iii) $\frac{x^2-7x+10}{(x-4)^2}\times\frac{x^2-7x+12}{x-5}$

Solution: (i) $\frac{5x+3}{5x-1}\times\frac{2x-1}{x+1}=\frac{(5x+3)(2x-1)}{(5x-1)(x+1)}$

$$=\frac{10x^2+x-3}{5x^2+4x-1}$$

(ii) $\frac{2x+1}{x-1}\times\frac{x-1}{x+3}=\frac{(2x+1)(x-1)}{(x-1)(x+3)}$

$=\frac{2x+1}{x+3}$ [Cancelling common factor $(x-1)$ from numerator and denominator]

(iii) $\frac{x^2-7x+10}{(x-4)^2}\times\frac{x^2-7x+12}{x-5}=\frac{(x^2-7x+10)(x^2-7x+12)}{(x-4)^2(x-5)}$

$$=\frac{(x-2)(x-5)(x-3)(x-4)}{(x-4)^2(x-5)}$$

$$= \frac{(x-2)(x-3)}{(x-4)}$$

[Cancelling common factor (x–4) (x – 5) from numerator and denominator]

$$= \frac{x^2 - 5x + 6}{x - 4}$$

Note: The result (product) obtained after cancelling the HCF from its numerator and denominator is called the result (product) **in lowest terms** or **in lowest form.**

You are also familiar with the division of a rational number, say, $\frac{2}{3}$ by a rational number, say, $\frac{5}{7}$ is given as $\frac{2}{3} \div \frac{5}{7} = \frac{2}{3} \times \frac{7}{5}$ where $\frac{7}{5}$ is the reciprocal of $\frac{5}{7}$. Similarly, division of a rational expression $\frac{P}{Q}$ by a non-zero rational expression $\frac{R}{S}$ is given by $\frac{P}{Q} \div \frac{R}{S} = \frac{P}{Q} \times \frac{S}{R}$ where P, Q, R, S are polynomials and $\frac{S}{R}$ is the **reciprocal expression** of $\frac{R}{S}$.

Example 4.23: Find the reciprocal of each of the following rational expressions:

(i) $\frac{x^2 + 20}{x^3 + 5x + 6}$ (ii) $-\frac{2y}{y^2 - 5}$ (iii) $x^3 + 8$

Solution: (i) Reciprocal of $\frac{x^2 + 20}{x^3 + 5x + 6}$ is $\frac{x^3 + 5x + 6}{x^2 + 20}$

(ii) Reciprocal of $-\frac{2y}{y^2 - 5}$ is $-\frac{y^2 - 5}{2y} = \frac{5 - y^2}{2y}$

(iii) Since $x^3 + 8 = \frac{x^3 + 8}{1}$, the reciprocal of $x^3 + 8$ is $\frac{1}{x^3 + 8}$

Example 4.24: Divide:

(i) $\frac{x^2 + 1}{x - 1}$ by $\frac{x - 1}{x + 2}$

(ii) $\frac{x^2 - 1}{x^2 - 25}$ by $\frac{x^2 - 4x - 5}{x^2 + 4x - 5}$ and express the result in lowest form.

Solution: (i) $\dfrac{x^2+1}{x-1} \div \dfrac{x-1}{x+2} = \dfrac{x^2+1}{x-1} \times \dfrac{x+2}{x-1}$

$$= \frac{(x^2+1)(x+2)}{(x-1)^2} = \frac{x^3+2x^2+x+2}{x^2-2x+1}$$

(ii) $\dfrac{x^2-1}{x^2-25} \div \dfrac{x^2-4x-5}{x^2+4x-5} = \dfrac{(x^2-1)(x^2+4x-5)}{(x^2-25)(x^2-4x-5)}$

$$= \frac{(x-1)(x+1)(x+5)(x-1)}{(x-5)(x+5)(x+1)(x-5)}$$

$$= \frac{(x-1)(x-1)}{(x-5)(x-5)}$$

[Cancelling HCF (x+1)(x+5)]

$$= \frac{x^2-2x+1}{x^2-10x+25}$$

The result $\dfrac{x^2-2x+1}{x^2-10x+25}$ is in lowest form.

CHECK YOUR PROGRESS 4.9

1. Find the product and express the result in lowest terms:

(i) $\dfrac{7x+2}{2x^2+3x+1} \times \dfrac{x+1}{7x^2-5x-2}$ (ii) $\dfrac{x^3+1}{x^4+1} \times \dfrac{x^3-1}{x^4-1}$

(iii) $\dfrac{3x^2-15x+18}{2x-4} \times \dfrac{17x+3}{x^2-6x+9}$ (iv) $\dfrac{5x-3}{5x+2} \times \dfrac{x+2}{x+6}$

(v) $\dfrac{x^2+1}{x-1} \times \dfrac{x+1}{x^2-x+1}$ (vi) $\dfrac{x^3+1}{x-1} \times \dfrac{x-1}{2x}$

(vii) $\dfrac{x-3}{x-4} \times \dfrac{x^2-5x+4}{x^2-2x-3}$ (viii) $\dfrac{x^2-7x+12}{x^2-2x-3} \times \dfrac{x^2-2x-24}{x^2-16}$

2. Find the reciprocal of each of the following rational expressions:

(i) $\dfrac{x^2+2}{x-1}$ (ii) $-\dfrac{3a}{1-a}$

(iii) $-\dfrac{7}{1-2x-x^2}$ (iv) x^4+1

3. Divide and express the result as a rational expression in lowest terms:

(i) $\dfrac{x^2+11x+18}{x^2-4x-117} \div \dfrac{x^2+7x+10}{x^2-12x-13}$ (ii) $\dfrac{6x^2+x-1}{2x^2-7x-15} \div \dfrac{4x^2+4x+1}{4x^2-9}$

(iii) $\dfrac{x^2+x+1}{x^2-9} \div \dfrac{x^3-1}{x^2-4x+3}$ (iv) $\dfrac{x^2+2x-24}{x^2-x-12} \div \dfrac{x^2-x-6}{x^2-9}$

(v) $\dfrac{3x^2+14x-5}{x^2-3x+2} \div \dfrac{3x^2+2x-1}{3x^2-3x-2}$ (vi) $\dfrac{2x^2+x-3}{(x-1)^2} \div \dfrac{2x^2+5x+3}{x^2-1}$

LET US SUM UP

- Special products, given below, occur very frequently in algebra:

  (i) $(x+y)^2 = x^2 + 2xy + y^2$ (ii) $(x-y)^2 = x^2 - 2xy + y^2$

  (iii) $(x+y)(x-y) = x^2 - y^2$ (iv) $(x+a)(x+b) = x^2 + (a+b)x + ab$

  (v) $(ax+b)(cx+d) = acx^2 + (ad+bc)x + bd$

  (vi) $(x+y)^3 = x^3 + 3xy(x+y) + y^3$ (vii) $(x-y)^3 = x^3 - 3xy(x-y) - y^3$

  (viii) $(x+y)(x^2-xy+y^2) = x^3 + y^3$ (ix) $(x-y)(x^2+xy+y^2) = x^3 - y^3$

- Factorization of a polynomial is a process of writing the polynomial as a product of two (or more) polynomials. Each polynomial in the product is called a factor of the given polynomial.

- A polynomial is said to be completely factorised if it is expressed as a product of factors, which have no factor other than itself, its negative, 1 or –1.

- Apart from the factorization based on the above mentioned special products, we can factorise a polynomial by taking monomial factor out which is common to some or all of the terms of the polynomial using distributive laws.

- HCF of two or more given polynomials is the product of the polynomial of the highest degree and greatest numerical coefficient each of which is a factor of each of the given polynomials.

- LCM of two or more given polynomials is the product of the polynomial of the lowest degree and the smallest numerical coefficient which are multiples of corresponding elements of each of the given polynomials.

- An algebraic expression, which can be expressed in the form $\frac{P}{Q}$ where P and Q are polynomials, Q being a non-zero polynomial, is called a rational expression.
- Operations on rational expressions are performed in the way, they are performed in case of rational numbers. Sum, Difference, Product and Quotient of two rational expressions are also rational expressions.
- Expressing a rational expression into lowest terms means cancellation of common factor, if any, from the numerator and denominator of the rational exprssion.

TERMINAL EXERCISE

1. Mark a tick ☑ against the correct alternative:

(i) If $120^2 - 20^2 = 25p$, then p is equal to

(A) 16 (B) 140 (C) 560 (D) 14000

(ii) $(2a^2 + 3)^2 - (2a^2 - 3)^2$ is equal to

(A) $24a^2$ (B) $24a^4$ (C) $72a^2$ (D) $72a^4$

(iii) $(a^2 + b^2)^2 + (a^2 - b^2)^2$ is equal to

(A) $2(a^2 + b^2)$ (B) $4(a^2 + b^2)$

(C) $4(a^4 + b^4)$ (D) $2(a^4 + b^4)$

(iv) If $m - \frac{1}{m} = -\sqrt{3}$, then $m^3 - \frac{1}{m^3}$ is equal to

(A) 0 (B) $6\sqrt{3}$ (C) $-6\sqrt{3}$ (D) $-3\sqrt{3}$

(v) $\frac{327 \times 327 - 323 \times 323}{327 + 323}$ is equal to

(A) 650 (B) 327 (C) 323 (D) 4

(vi) $8m^3 - n^3$ is equal to:

(A) $(2m - n)(4m^2 - 2mn + n^2)$ (B) $(2m - n)(4m^2 + 2mn + n^2)$

(C) $(2m - n)(4m^2 - 4mn + n^2)$ (D) $(2m - n)(4m^2 + 4mn + n^2)$

(vii) $\frac{467 \times 467 \times 467 + 533 \times 533 \times 533}{467 \times 467 - 467 \times 533 + 533 \times 533}$ is equal to

(A) 66 (B) 198 (C) 1000 (D) 3000

(viii) The HCF of $36a^5b^2$ and $90a^3b^4$ is

(A) $36a^3b^2$ (B) $18a^3b^2$

(C) $90a^3b^4$ (D) $180a^5b^4$

(ix) The LCM of $x^2 - 1$ and $x^2 - x - 2$ is

(A) $(x^2 - 1)(x - 2)$ (B) $(x^2 - 1)(x + 2)$

(C) $(x - 1)^2 (x + 2)$ (D) $(x + 1)^2 (x - 2)$

(x) Which of the following is not a rational expression?

(A) $\sqrt{33}$ (B) $x + \dfrac{1}{\sqrt{5x}}$

(C) $8\sqrt{x} + 6\sqrt{y}$ (D) $\dfrac{x - \sqrt{3}}{x + \sqrt{3}}$

2. Find each of the following products:

(i) $(a^m + a^n)(a^m - a^n)$ (ii) $(x + y + 2)(x - y + 2)$

(iii) $(2x + 3y)(2x + 3y)$ (iv) $(3a - 5b)(3a - 5b)$

(v) $(5x + 2y)(25x^2 - 10xy + 4y^2)$ (vi) $(2x - 5y)(4x^2 + 10xy + 25y^2)$

(vii) $\left(a + \dfrac{5}{4}\right)\left(a + \dfrac{4}{5}\right)$ (viii) $(2z^2 + 3)(2z^2 - 5)$

(ix) $99 \times 99 \times 99$ (x) $103 \times 103 \times 103$

(xi) $(a + b - 5)(a + b - 6)$ (xii) $(2x + 7z)(2x + 5z)$

3. If $x = a - b$ and $y = b - c$, show that

$(a - c)(a + c - 2b) = x^2 - y^2$

4. Find the value of $64x^3 - 125z^3$ if $4x - 5z = 16$ and $xz = 12$.

5. Factorise:

(i) $x^7 y^6 + x^{22} y^{20}$ (ii) $3a^5b - 243ab^5$

(iii) $3a^6 + 12a^4b^2 + 12a^2b^4$ (iv) $a^4 - 8a^2b^3 + 16b^6$

(v) $3x^4 + 12y^4$ (vi) $x^8 + 14x^4 + 81$

(vii) $x^2 + 16x + 63$ (viii) $x^2 - 12x + 27$

(ix) $7x^2 + xy - 6y^2$ (x) $5x^2 - 8x - 4$

(xi) $x^6 - 729y^6$ (xii) $125a^6 + 64b^6$

6. Find the HCF of

(i) $x^3 - x^5$ and $x^4 - x^7$

(ii) $30(x^2 - 3x + 2)$ and $50(x^2 - 2x + 1)$

7. Find the LCM of

(i) $x^3 + y^3$ and $x^2 - y^2$

(ii) $x^4 + x^2y^2 + y^4$ and $x^2 + xy + y^2$

8. Perform the indicated operation:

(i) $\dfrac{x+1}{(x-1)^2} + \dfrac{1}{x+1}$

(ii) $\dfrac{2x^2+2x-7}{x^2+x-6} - \dfrac{x-1}{x-2}$

(iii) $\dfrac{x-1}{x-2} \times \dfrac{3x+1}{x^2-4}$

(iv) $\dfrac{x^2-1}{x^2-25} \div \dfrac{x^2-4x-5}{x^2+4x-5}$

9. Simpify: $\dfrac{2}{a-1} - \dfrac{2}{a+1} - \dfrac{4}{a^2+1} - \dfrac{8}{a^4+1}$

[Hint : $\dfrac{2}{a-1} - \dfrac{2}{a+1} = \dfrac{4}{a^2-1}$; now combine next term and so on]

10. If $m = \dfrac{x+1}{x-1}$ and $n = \dfrac{x-1}{x+1}$, find $m^2 + n^2 - mn$.

ANSWERS TO CHECK YOUR PROGRESS

4.1

1. (i) $25x^2 + 20xy + y^2$ (ii) $x^2 - 6x + 9$ (iii) $a^2b^2 + 2abcd + c^2d^2$

(iv) $4x^2 - 20xy + 5y^2$ (v) $\dfrac{x^2}{9} + \dfrac{2}{3}x + 1$ (vi) $\dfrac{z^2}{4} - \dfrac{1}{3}z + \dfrac{1}{9}$

(vii) $a^4 - 25$ (viii) $x^2y^2 - 1$ (ix) $x^2 + \dfrac{25}{12}x + 1$

(x) $\frac{4}{9}x^4 - \frac{25}{9}x^2 - 1$ (xi) $6x^2 + 13xy + 6y^2$ (xii) $21x^2 + 8xy - 5y^2$

2. (i) $40x^2$ (ii) $2a^6 + 18$ (iii) $2(a^2x^2 + b^2y^2)$ (iv) $32p^2q^2$

3. (i) 10404 (ii) 11664 (iii) 4761 (iv) 996004
   (v) 6384 (vi) 22451 (vii) 89964 (viii) 249936
   (ix) 11445 (x) 5621 (xi) 8930 (xii) 989028

4.2

1. (i) $27x^3 + 36x^2y + 36xy^2 + 64y^3$ (ii) $p^3 - 3p^2qr + 3pq^2r^2 - q^3r^3$

   (iii) $a^3 + a^2b + \frac{ab^2}{3} + \frac{b^3}{27}$ (iv) $\frac{a^3}{27} - \frac{a^2b}{3} + ab^2 - b^3$

   (v) $\frac{a^6}{8} + \frac{1}{2}a^4b^2 + \frac{2}{3}a^2b^4 + \frac{8}{27}b^6$ (vi) $\frac{a^6x^9}{27} - \frac{2}{3}a^4b^3x^6y^2 + 4a^2b^6x^3y^4 - 8b^9y^6$

2. (i) 512 (ii) 1728 (iii) 5832 (iv) 12167 (v) 148877
   (vi) 110592 (vii) 357911 (viii) 328509 (ix) 912663 (x) 970299

3. (i) $8x^3 + y^3$ (ii) $x^3 - 8$ (iii) $x^3 + 1$

   (iv) $8y^3 - 27z^6$ (v) $64x^3 + 27y^3$ (vi) $27x^3 - \frac{1}{343}y^3$

4. (i) 100 (ii) 1115

5. (i) 15616 (ii) $\frac{27027}{125}$

6. (i) $120x^2 + 250$ (ii) $1000y^3$ (iii) $19x^3 - 19y^3$ (iv) $-117x^3 - 126$

7. (i) 1000 (ii) 444

4.3

1. $5x(2y - 3z)$
2. $abc\,(c - b)$
3. $3p(2p - 5q + 9)$
4. $(b - c)(a^2 - b)$
5. $(4x - y)^2 (8ax - 2ay - b)$
6. $x(x + y)(x^2 - xy + y^2)$
7. $25(2 + 5p)(2 - 5p)$
8. $(1 + 16y^4)(1 + 4y^2)(1 + 2y)(1 - 2y)$
9. $(5x + 1)(1 - x)$
10. $(a^2 + bc + ab + ac)(a^2 + bc - ab - ac)$

11. $(5x + 6y - 1)(5x - 6y - 1)$ 12. $(7x - y + 1)(7x - y - 1)$

13. $(m + 7)^2$ 14. $(2x - 1)^2$

15. $(6a + 5)^2$ 16. $(x^3 - 4)^2$

17. $(a^4 + 7a^2 + 1)(a^2 + 3a + 1)(a^2 - 3a + 1)$

18. $(2a^2 + 6ab + 9b^2)(2a^2 - 6ab + 9b^2)$

19. $(x^2 + 2x + 2)(x^2 - 2x + 2)$

20. $(3a^2 + 5a + 4)(3a^2 - 5a + 4)$ 21. (i) 40 (ii) 57200

4.4

1. $(a + 6b)(a^2 - 6ab + 36b^2)$ 2. $(a - 7)(a^2 + 7a + 49)$

3. $(x + 4y)^3$ 4. $(2x - 3y)^3$

5. $(2x - 5y)^3$ 6. $(4k - 3)^3$

7. $(9x^2 - 2)(81x^4 + 18x^2 + 4)$ 8. $x^2(1 + y^2)(1 - y^2 + y^4)$

9. $2a(2a^2 - 3b^2)(4a^2 + 6a^2b^2 + 9b^4)$ 10. $(3b - a - 1)(9b^2 + 3ab + 3b + a^2 + a + 1)$

11. $(2a - 3b + 4c)(4a^2 + 9b^2 - 6ab - 8ac + 12bc + 16c^2$

12. $(4x - 2y + 1)(16x^2 + 8xy - 4x + 4y^2 - 4y + 1)$

4.5

1. $(x + 3)(x + 8)$ 2. $(x - 6y)(x - 9y)$ 3. $(x + 3)(2x - 1)$

4. $2(x - 2y)(3x + y)$ 5. $(2x^2 + 1)(x + 1)(x - 1)$ 6. $(x + 15y)(x - 2y)$

7. $(x + 2)(2x + 7)$ 8. $(2y - 3)(5y - 2)$ 9. $(x - 1)(2x + 1)$

10. $(12 - m)(m + 9)$ 11. $(2a - b - 6)(2a - b + 5)$ 12. $(9y - 7)(5x + y)$

4.6

1. (i) $3xy^2$ (ii) $12y^3x^5$ (iii) $(x + 1)^2$ (iv) $x + 2$ (v) $6(x + 2)$

(vi) $(x + 5)^2$ (vii) $(2x - 5)^2$ (viii) $x^2 - 1$ (ix) $x - y$ (x) $6(x - 1)$

2. (i) $75x^3y^2$ (ii) $240x^3y^4$ (iii) $(x - 1)(x + 1)^3$

(iv) $x^2 + 4x + 4$ (v) $72(x + 2)^3(x^2 - 2x + 4)$ (vi) $(x + 1)^2(x + 5)^3$

(vii) $(x - 4)(x + 4)^2(2x - 5)^3$ (viii) $x^4 - 1$ (ix) $(x - 1)(x + 1)(x^2 + x + 1)$

(x) $18(x - 1)(x - 2)(x - 3)$

4.7

1. (i), (ii), (iii), (v), (vii) and (viii)

4.8

1. (i) $\frac{2x^2}{x-2}$ (ii) $\frac{2x^2+2x-7}{x^2+x-6}$ (iii) $\frac{2x^2+2}{x^3-x^2-x+1}$

(iv) $\frac{4x^2+5x+28}{x^3+4x^2-16x+64}$ (v) $\frac{2x}{x+3}$ (vi) $\frac{2x^2+8}{x^2-4}$

(vii) $\frac{2x^3+3x^2-1}{x^3+2x^2+x+2}$ (viii) $\frac{5}{6x^2}$

2. (i) $\frac{x-6}{x^2-4}$ (ii) $\frac{8x}{4x^2-1}$ (iii) $\frac{x^2-1}{x}$

(iv) $\frac{2-x}{x^2-x}$ (v) $\frac{x^2+2}{x-4}$ (vi) $\frac{2x^3+1}{(x^2+2)^2}$

(vii) $\frac{x^2-15x+16}{2(x^3+3x^2-9x-27)}$ (viii) $\frac{1-x}{1+x}$

3. (i) 2 (ii) 6 (iii) 2 (iv) 110 (v) $8\sqrt{15}$

(vi) 115 (vii) 0 (viii) 18 (ix) ± 5 (x) 14

4.9

1. (i) $\frac{1}{2x^2-x-1}$ (ii) $\frac{x^4+x^2+1}{x^6+x^4+x^2+1}$ (iii) $\frac{51x+9}{2x-6}$

(iv) $\frac{5x^2+7x-6}{5x^2+32x+12}$ (v) $\frac{x^3+x^2+x+1}{x^3-2x^2+2x-1}$ (vi) $\frac{x^3+1}{2x}$

(vii) $\frac{x-1}{x+1}$ (viii) $\frac{x-6}{x+1}$

2. (i) $\frac{x-1}{x^2+2}$ (ii) $\frac{a-1}{3a}$ (iii) $\frac{x^2+2x-1}{7}$ (iv) $\frac{1}{x^4+1}$

3. (i) $\frac{x+1}{x+5}$ (ii) $\frac{6x^2-11x+3}{2x^2-9x-5}$ (iii) $\frac{1}{x+3}$

(iv) $\frac{x+6}{x+2}$ (v) $\frac{2x^2+11x+5}{x^2-1}$ (vi) 1

ANSWERS TO TERMINAL EXERCISE

1. (i) C (ii) A (iii) D (iv) A (v) D (vi) B (vii) C (viii) B (ix) A (x) C

2. (i) $a^{2m} - a^{2n}$ (ii) $x^2 - y^2 + 4x + 4$ (iii) $4x^2 + 12xy + 9y^2$

(iv) $9a^2 - 30ab + 25b^2$ (v) $125x^3 + 8y^3$ (vi) $8x^3 - 125y^3$

(vii) $a^2 + \frac{41}{20}a + 1$ (viii) $4z^4 - 4z^2 - 15$ (ix) 970299

(x) 1092727 (xi) $a^2 + 2ab - 11a + 30$ (xii) $4x^2 + 24xz + 35z^2$

4. 15616

5. (i) $x^7y^6(1 + x^{15}y^{14})$ (ii) $3ab(a - 3b)(a + 3b)(a^2 + 9b^2)$

(iii) $3a^2(a^2 + 2b^2)^2$ (iv) $(a^2 - 4b^3)^2$

(v) $3(x^2 + 2xy + 2y^2)$ (vi) $(x^4 - 2x^2 + 9)(x^4 + 2x^2 + 9)$

(vii) $(x+9)(x+7)$ (viii) $(x-3)(x-9)$

(ix) $(x+y)(7x-6y)$ (x) $(x-2)(5x+2)$

(xi) $(x-3y)(x+3y)(x^2-3xy+9y^2)(x^2+3xy+9y^2)$

(xii) $(5a^2+4b^2)(25a^4-20a^2b^2+16b^4)$

6. (i) $x^3(1-x)$ (ii) $10(x-1)$

7. (i) $(x^2-y^2)(x^2-xy+y^2$ (ii) $x^4+x^2y^2+y^4$

8. (i) $\frac{2x^2+2}{x^3-x^2-x+1}$ (ii) $\frac{x+2}{x+3}$

(iii) $\frac{3x^2-2x-1}{x^3+2x^2-4x-8}$ (iv) $\frac{x^2-2x+1}{x^2-10x+25}$

9. $\frac{16}{a^8-1}$

10. $\frac{x^4+14x^2+1}{x^4-2x^2+1}$

5

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LINEAR EQUATIONS

You have learnt about basic concept of a variable and a constant. You have also learnt about algebraic exprssions, polynomials and their zeroes. We come across many situations such as six added to twice a number is 20. To find the number, we have to assume the number as x and formulate a relationship through which we can find the number. We shall see that the formulation of such expression leads to an equation involving variables and constants. In this lesson, you will study about linear equations in one and two variables. You will learn how to formulate linear equations in one variable and solve them algebraically. You will also learn to solve linear equations in two variables using graphical as well as algebraic methods.

OBJECTIVES

After studying this lesson, you will be able to

- *identify linear equations from a given collection of equations;*
- *cite examples of linear equations;*
- *write a linear equation in one variable and also give its solution;*
- *cite examples and write linear equations in two variables;*
- *draw graph of a linear equation in two variables;*
- *find the solution of a linear equation in two variables;*
- *find the solution of a system of two linear equations graphically as well as algebraically;*
- *Translate real life problems in terms of linear equations in one or two variables and then solve the same.*

EXPECTED BACKGROUND KNOWLEDGE

- Concept of a variable and constant

- Algebraic expressions and operations on them
- Concept of a polynomial, zero of a polynomial and operations on polynomials

5.1 LINEAR EQUATIONS

You are already familiar with the algebraic expressions and polynomials. The value of an algebraic expression depends on the values of the variables involved it. You have also learnt about polynomial in one variable and their degrees. A polynomials in one variable whose degree is one is called a **linear polynomial** in one variable. When two expressions are separated by an equality sign, it is called an **equation**. Thus, in an equation, there is always an equality sign. The equality sign shows that the expression to the left of the sign (the left had side or LHS) is equal to the expression to the right of the sign (the right hand side or RHS). For example,

$$3x + 2 = 14 \qquad ...(1)$$

$$2y - 3 = 3y + 4 \qquad ...(2)$$

$$z^2 - 3z + 2 = 0 \qquad ...(3)$$

$$3x^2 + 2 = 1 \qquad ...(4)$$

are all equations as they contain equality sign and also contain variables. In (1), the LHS = $3x + 2$ and RHS = 14 and the variable involved is x. In (2), LHS = $2y - 3$, RHS = $3y + 4$ and both are linear polynomials in one variable. In (3) and (4), LHS is a polynomial of degree two and RHS is a number.

You can also observe that in equation (1), LHS is a polynomial of degree one and RHS is a number. In (2), both LHS and RHS are linear polynomials and in (3) and (4), LHS is a quadratic polynomial. The equations (1) and (2) are linear equations and (3) and (4) are not linear equations.

In short, an equation is a condition on a variable. The condition is that two expressions, i.e., LHS and RHS should be equal. It is to be noted that atleast one of the two expressions must contain the variable.

It should be noted that the equation $3x - 4 = 4x + 6$ is the same as $4x + 6 = 3x - 4$. Thus, an equation remains the same when the expressions on LHS and RHS are interchanged. This property is often use in solving equations.

An equation which contains two variables and the exponents of each variable is one and has no term involving product of variables is called a linear equation in two variables. For example, $2x + 3y = 4$ and $x - 2y + 2 = 3x + y + 6$ are linear equations in two variables. The equation $3x^2 + y = 5$ is not a linear equation in two variables and is of degree 2, as the exponent of the variable x is 2. Also, the equation $xy + x = 5$ is not a linear equation in two variables as it contains the term xy which is the product of two variables x and y.

The general form of a linear equation in one variable is $ax + b = 0$, $a \neq 0$, a and b are constants. The general form of a linear equation in two variables is $ax + by + c = 0$ where a, b and c are real numbers such that at atleast one of a and b is non-zero.

Example 5.1: Which of the following are linear equations in one variable? Also write their LHS and RHS.

(i) $2x + 5 = 8$

(ii) $3y - z = y + 5$

(iii) $x^2 - 2x = x + 3$

(iv) $3x - 7 = 2x + 3$

(v) $2 + 4 = 5 + 1$

Solution:

(i) It is a linear equation in x as the exponent of x is 1. LHS = $2x + 5$ and RHS = 8

(ii) It is not a linear equation in one variable as it contains two variables y and z. Here, LHS = $3y - z$ and RHS = $y + 5$

(iii) It is not a linear equation as highest exponent of x is 2. Here, LHS = $x^2 - 2x$ and RHS = $x + 3$.

(iv) It is a linear equation in x as the exponent of x in both LHS and RHS is one.
LHS = $3x - 7$, RHS = $2x + 3$

(v) It is not a linear equation as it does not contain any variable. Here LHS = $2 + 4$ and RHS = $5 + 1$.

Example 5.2: Which of the following are linear equations in two variables.

(i) $2x + z = 5$

(ii) $3y - 2 = x + 3$

(iii) $3t + 6 = t - 1$

Solution:

(i) It is a linear equation in two variables x and z.

(ii) It is a linear equation in two variables y and x.

(iii) It is not a linear equation in two variables as it contains only one variable t.

CHECK YOUR PROGRESS 5.1

1. Which of the following are linear equations in one variable?

   (i) $3x - 6 = 7$

   (ii) $2x - 1 = 3z + 2$

(iii) $5 - 4 = 1$

(iv) $y^2 = 2y - 1$

2. Which of the following are linear equations in two variables:

(i) $3y - 5 = x + 2$

(ii) $x^2 + y = 2y - 3$

(iii) $x + 5 = 2x - 3$

5.2 FORMATION OF LINEAR EQUATIONS IN ONE VARIABLE

Consider the following situations:

(i) 4 more than *x* is 11

(ii) A number *y* divided by 7 gives 2.

(iii) Reena has some apples with her. She gave 5 apples to her sister. If she is left with 3 apples, how many apples she had.

(iv) The digit at tens place of a two digit number is two times the digit at units place. If digits are reversed, the number becomes 18 less than the original number. What is the original number?

In (i), the equation can be written as $x + 4 = 11$. You can verify that $x = 7$ satisfies the equation. Thus, $x = 7$ is a solution.

In (ii), the equation is $\frac{y}{7} = 2$.

In (iii), You can assume the quantity to be found out as a variable say x, i.e., let Reena has x apples. She gave 5 apples to her sister, hence she is left with $x - 5$ apples. Hence, the required equation can be written as $x - 5 = 3$, or $x = 8$.

In (iv), Let the digit in the unit place be x. Therefore, the digit in the tens place should be 2x. Hence, the number is

$$10\,(2x) + x = 20x + x = 21x$$

When the digit are reversed, the tens place becomes x and unit place becomes 2x. Therefore, the number is $10x + 2x = 12x$. Since original number is 18 more than the new number, the equation becomes

$$21x - 12x = 18$$

or $$9x = 18$$

CHECK YOUR PROGRESS 5.2

Form a linear equation using suitable variables for the following situations:

1. Twice a number subtracted from 15 is 7.
2. A motor boat uses 0.1 litres of fuel for every kilometer. One day, it made a trip of x km. Form an equation in x, if the total consumption of fuel was 10 litres.
3. The length of rectangle is twice its width. The perimeter of rectangle is 96m. [Assume width of rectangle as y m]
4. After 15 years, Salma will be four times as old as she is now. [Assume present age of Salma as t years]

5.3 SOLUTION OF LINEAR EQUATIONS IN ONE VARIABLE

Let us consider the following linear equation in one variable,

$$x - 3 = -2$$

Here LHS = $x - 3$ and RHS = -2

Now, we evaluate RHS and LHS for some values of x

x	LHS	RHS
0	-3	-2
1	-2	-2
3	0	-2
4	1	-2

We observe that LHS and RHS are equal only when $x = 1$. For all other values of x, LHS $\neq$ RHS. We say that the value of x equal to 1 **satisfies** the equation or **$x = 1$ is a solution of the equation**.

A number, which when substituted for the variable in the equation makes LHS equal to RHS, is called its solution. We can find the solution of an equation by trial and error method by taking different values of the variable. However, we shall learn a systematic way to find the solution of a linear equation.

An equation can be compared with a balance for weighing, its sides are two pans and the equality symbol '=' tells us that the two pans are in balance.

We have seen the working of balance, If we put equal (and hence add) or remove equal weights, (and hence subtract) from both pans, the two pans remain in balance. Thus we can translate for an equation in the following way:

1. Add same number to both sides of the equation.
2. Subtract same number from both sides of the equation.
3. Multiply both sides of the equation by the same non-zero number.
4. Divide both sides of the equation by the same non-zero number.

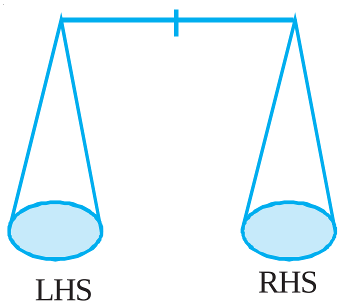

Fig 5.1

We now consider some examples:

Example 5.3: Solve $5 + x = 8$.

Solution: Subtracting 5 from both sides of the equation.

We get $\quad 5 + x - 5 = 8 - 5$

or $\quad x + 0 = 3$

or $\quad x = 3$

So, $x = 3$ is the solution of the given equation.

Check: When $x = 3$, LHS $= 5 + x = 5 + 3 = 8$ and R.H.S. $= 8$.
Therefore, LHS = RHS.

Example 5.4: Solve: $y - 2 = 7$.

Solution: Adding 2 to both sides of the equation, we get

$$y - 2 + 2 = 7 + 2$$

or $\quad y = 9$

Hence, $y = 9$ is the solution.

Check: When $y = 9$, LHS $= y - 2 = 9 - 2 = 7$ and RHS $= 7$. Therefore, LHS = RHS.

Example 5.5: Solve: $7x + 2 = 8$.

Solution: Subtracting 2 from both sides of the equation, we get

$$7x + 2 - 2 = 8 - 2$$

or $\quad 7x = 6$

or $\quad \dfrac{7x}{7} = \dfrac{6}{7}$ (dividing both sides by 7)

or $\quad x = \dfrac{6}{7}$

Therefore, $x = \dfrac{6}{7}$ is the solution of the equation.

Example 5.6: Solve: $\frac{3y}{2} - 3 = 9$

Solution: Adding 3 to both sides of the equation, we get

$$\frac{3y}{2} - 3 + 3 = 9 + 3$$

or $\qquad \frac{3y}{2} = 12$

or $\qquad \frac{3y}{2} \times 2 = 12 \times 2$ (Multiplying both sides by 2)

or $\qquad 3y = 24$

or $\qquad \frac{3y}{3} = \frac{24}{3}$ (Dividing both sides by 3)

or $\qquad y = 8$

Hence, $y = 8$ is the solution.

Example 5.7: Solve the equation $2(x + 3) = 3(2x - 7)$

Solution: The equation can be written as

	$2x + 6 = 6x - 21$	
or	$6x - 21 = 2x + 6$	[Interchanging LHS and RHS]
or	$6x - 21 + 21 = 2x + 6 + 21$	[Adding 21 on both sides]
or	$6x = 2x + 27$	
or	$6x - 2x = 2x + 27 - 2x$	[Subtracting 2x from both sides]
or	$4x = 27$	
or	$x = \frac{27}{4}$	

Thus, $x = \frac{27}{4}$ is the solution of the equation.

Note:

1. It is not necessary to write the details of what we are adding, subtracting, multiplying or dividing each time.
2. The process of taking a term from LHS to RHS or RHS to LHS, is called transposing.
3. When we transpose a term from one side to other side, sign '+' changes to '–', '–' to '+'.

4. A linear equation in one variable can be written as ax + b = 0, where a and b are constants and x is the variable. Its solution is $x = -\frac{b}{a},\ a \neq 0$.

Example 5.8: Solve 3x – 5 = x +3

Solution: We have 3x – 5 = x + 3

or 3x = x + 3 + 5

or 3x – x = 8

or 2x = 8

or x = 4

Therefore, x = 4 is the solution of the given equation.

CHECK YOUR PROGRESS 5.3

Solve the following equations:

1. x – 5 = 8
2. 19 = 7 + y
3. 3z + 4 = 5z + 4
4. $\frac{1}{3}y + 9 = 12$
5. 5(x – 3) = x + 5

5.4 WORD PROBLEMS

You have learnt how to form linear equations in one variable. We will now study some applications of linear equations.

Example 5.9: The present age of Jacob's father is three times that of Jacob. After 5 years, the difference of their ages will be 30 years. Find their present ages.

Solution: Let the present age of Jacob be x years.

Therefore, the present age of his father is 3x years.

After 5 years, the age of Jacob = (x + 5) years.

After 5 years, the age of his father = (3x + 5) years.

The difference of their ages = (3x + 5) – (x + 5) years, which is given to be 30 years, therefore

$$3x + 5 - (x + 5) = 30$$

or $\quad 3x + 5 - x - 5 = 30$

or $\quad 3x - x = 30$

or $\quad 2x = 30$

or $\quad x = 15$

Therefore, the present age of Jacob is 15 years and the present age of his father = 3x = 3 × 15 = 45 years.

Check: After 5 years, age of Jacob = 15 + 5 = 20 years

After 5 years, age of his father = 45 + 5 =50 years

Difference of their ages = 50 – 20 = 30 years

Example 5.10 : The sum of three consecutive even integers is 36. Find the integers.

Solution: Let the smallest integer be x.

Therefore, other two integers are x + 2 and x + 4.

Since, their sum is 36, we have

$$x + (x + 2) + (x + 4) = 36$$

or $\quad 3x + 6 = 36$

or $\quad 3x = 36 - 6 = 30$

or $\quad x = 10$

Therefore, the required integers are 10, 12 and 14.

Example 5.11: The length of a rectangle is 3 cm more than its breadth. If its perimeter is 34 cm find its length and breadth.

Solution: Let the breadth of rectangle be x cm

Therefore, its length = x + 3

Now, since perimeter = 34 cm

We have $\quad 2(x + 3 + x) = 34$

or $\quad 2x + 6 + 2x = 34$

or $\quad 4x = 34 - 6$

or $\quad 4x = 28$

or $\quad x = 7$

Therefore, breadth = 7 cm, and length = 7 + 3 = 10 cm.

CHECK YOUR PROGRESS 5.4

1. The sum of two numbers is 85. If one number exceeds the other by 7, find the numbers.
2. The age of father is 20 years more than twice the age of the son. If sum of their ages is 65 years, find the age of the son and the father.
3. The length of a rectangle is twice its breadth. If perimeter of rectangle is 66 cm, find its length and breadth.
4. In a class, the number of boys is $\frac{2}{5}$ of the number of girls. Find the number of girls in the class, if the number of boys is 10.

5.5 LINEAR EQUATIONS IN TWO VARIABLES

Neha went to market to the purchase pencils and pens. The cost of one pencil is Rs 2 and cost of one pen is Rs 4. If she spent Rs 50, how many pencils and pens she purchased?

Since, we want to find the number of pencils and pens, let us assume that she purchased x pencils and y pens. Then,

Cost of x pencils = Rs 2 x

Cost of y pens = Rs 4 y

Since, total cost in Rs 50, we have

$$2x + 4y = 50 \qquad \text{...(1)}$$

This is a linear equation in two variables x and y as it is of the form $ax + by + c = 0$

We shall now take different values of x and y to find the solution of the equation (1)

1. If $x = 1$, $y = 12$, then LHS $= 2 \times 1 + 4 \times 12 = 2 + 48 = 50$ and RHS $= 50$. Therefore, $x = 1$ and $y = 12$ is a solution.
2. If $x = 3$, $y = 11$, then LHS $= 2 \times 3 + 4 \times 11 = 50$ and RHS $= 50$. Therefore, $x = 3$, $y = 11$ is also a solution.
3. If $x = 4$, $y = 10$, then LHS $= 9 \times 4 + 4 \times 10 = 48$ and RHS $= 50$. Therefore, $x = 4$, $y = 10$ is **not** a solution of the equation.

Thus, a linear equation in two variables has more than one solution.

We have seen that a linear equation in one variable 'x' is of the form $ax + b = 0$, $a \neq 0$. It has only one solution i.e., $x = -\frac{b}{a}$. However, a linear equation in two variables x and y is of the form

$$ax + by + c = 0 \qquad ...(1)$$

where a, b and c are constants and atleast one of a or b is non-zero. Let $a \neq 0$, then (1) can be written as

$$ax = -by - c$$

or $$x = -\frac{b}{a}y - \frac{c}{a}$$

Now, for each value of y, we get a unique value of x. Thus, a linear equation in two variables will have **infinitely many solutions**.

Note: A linear equation $ax + c = 0$, $a \neq 0$, can be considered as a linear equation in two variables by expressing it as

$$ax + 0y + c = 0$$

i.e., by taking the coefficient of y as zero. It still has many solutions such as

$$x = -\frac{c}{a},\ y = 0;\ \ x = -\frac{c}{a},\ y = 1 \text{ etc.}$$

i.e., for each value of y, the value of x will be equal to $-\frac{c}{a}$.

Example 5.12: The sum of two integers is 15. Form a linear equation in two variables.

Solution: Let the two integers be x and y. Therefore, their sum = x + y. It is given that the sum is 15.

Hence, required equation is $x + y = 15$.

Example 5.13: For the equation $4x - 5y = 2$, verify whether (i) $x = 3, y = 2$ and (ii) $x = 4, y = 1$ are solutions or not.

Solution: (i) We have $4x - 5y = 2$

When $x = 3, y = 2$, LHS $= 4x - 5y = 4 \times 3 - 5 \times 2$

$= 12 - 10 = 2$

$=$ RHS

Therefore, $x = 3, y = 2$ is a solution of the given equation.

(ii) When $x = 4, y = 1$, LHS $= 4 \times 4 - 5 \times 1 = 16 - 5 = 11$

But RHS = 2. Therefore, LHS $\neq$ RHS

Hence, $x = 4, y = 1$ is not a solution.

CHECK YOUR PROGRESS 5.5

1. Form linear equations in two variables using suitable variables for the unknowns.
   (i) The perimeter of a rectangle is 98 cm. [Take length as x and breadth as y.]
   (ii) The age of father is 10 years more than twice the age of son.
   (iii) A number is 10 more than the other number.
   (iv) The cost of 2kg apples and 3 kg oranges is Rs. 120. [Take x and y as the cost per kg of apples and oranges respectively.]

Write True or False for the following:

2. $x = 0, y = 3$ is a solution of the equation
   $3x + 2y - 6 = 0$
3. $x = 2, y = 5$ is a solution of the equation
   $5x + 2y = 10$

5.6 GRAPH OF A LINEAR EQUATION IN TWO VARIABLES

You will now learn to draw the graph of a linear equation in two variables. Consider the equation $2x + 3y = 12$. It can be written as

$$2x = 12 - 3y \text{ or } 3y = 12 - 2x$$

$$x = \frac{12-3y}{2} \quad \text{or} \quad y = \frac{12-2x}{3}$$

Now, for each value of y or for each value of x, we get a unique corresponding value of x or y. We make the following table for the values of x and y which satisfy the equation:

$$2x + 3y = 12$$

x	0	6	3	9	– 3
y	4	0	2	– 2	6

Thus, $x = 0, y = 4$; $x = 6, y = 0$; $x = 3, y = 2$; $x = 9, y = -2$; $x = -3, y = 6$ are all solutions of the given equation.

We write these solutions as order pairs (0, 4), (6, 0), (3, 2), (9, – 2) and (– 3, 6).

Here, first entry gives the value of x and the corresponding second entry gives the value of y. We will now learn to draw the graph of this equation by plotting these ordered pairs in a plane and then join them. In the graph of $2x + 3y = 12$, the points representing the solutions will be on a line and a point which is not a solution, will not lie on this line. Each point also called orderd pair, which lies on the line will give a solution and a point which does not lie on the line will not be a solution of the equation.

To draw the graph of a linear equation in two variables, we will first plot these points in a plane. We proceed as follows:

Step 1: We take two perpendicular lines X'OX and YOY' intersecting at O. Mark the real numbers on X'OX and YOY' by considering them as number lines with the point O as the real number 0 as shown in Fig 5.2. These two lines divide the plane into four parts, called first quadrant, second quadrant, third quadrant and fourth quadrant. The number line X'OX is called **x-axis** and the line Y'OY is called **y-axis**. Since, we have taken x-axis and y-axis, perpendicular to each other in a plane, we call the plane as coordinate plane or **cartesian plane** in the honour of French mathematician Descartes who invented this system to plot a point in the plane.

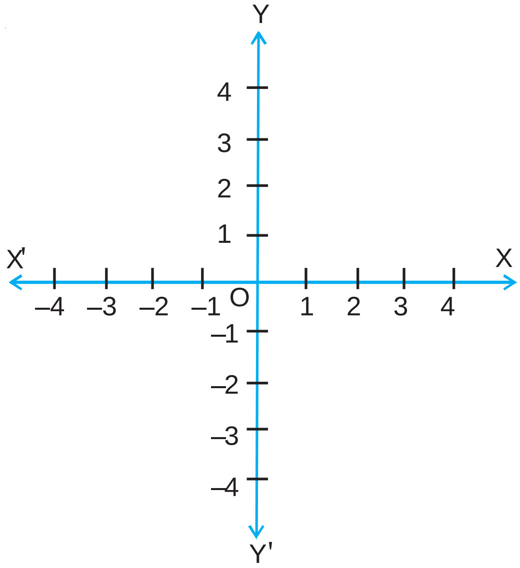

Fig 5.2

Step 2: To plot a point say (3, 2), take the point 3 on x-axis and through this point, draw a line '*l*' perpendicular to x-axis (i.e. parallel to y-axis). Now take the point 2 on y-axis and through 2, draw a line 'm' perpendicular to y-axis (i.e. parallel to x-axis) to meet *l* at P. The point P represents the point (3, 2) on the plane.

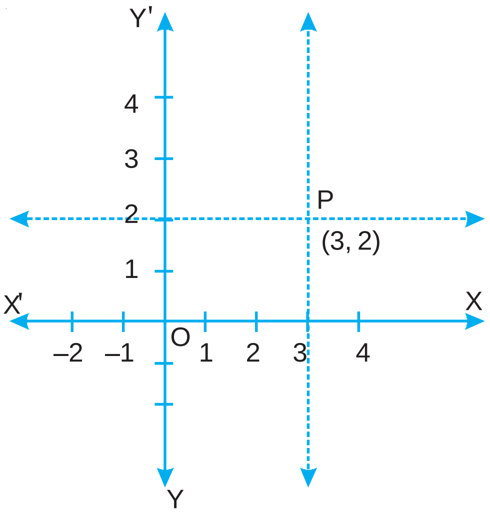

Fig. 5.3

Note 1: It may be noted that, for the ordered pair (a, b), a is called **x-coordinate** and b is called **y-coordinate**.

Note 2: Every point on x-axis can be written as (a, 0) i.e. its y-coordinate is zero and every point on y-axis is of the form (0, b) i.e., its x-coordinate is zero. The coordinates of the point O are (0, 0).

Note 3: In the first quadrant, both x and y coordinates are positive, in the second quadrant, x coordinate is negative and y coordinate is positive, in the third quadrant both x and y coordinates are negative and in the fourth quadrant, x-coordinate is positive and y-coordinate is negative.

Example 5.14: Represent the point (–2, 3) in the coordinate plane.

Solution: Draw x-axis and y-axis on the plane and mark the points on them. Take the point –2 on x-axis and draw the line *l* parallel to y-axis. Now take the point 3 on y-axis and draw the line 'm' parallel to x-axis to meet *l* at P. The point P represent (–2, 3), we say (–2 , 3) are coordinates of the point P.

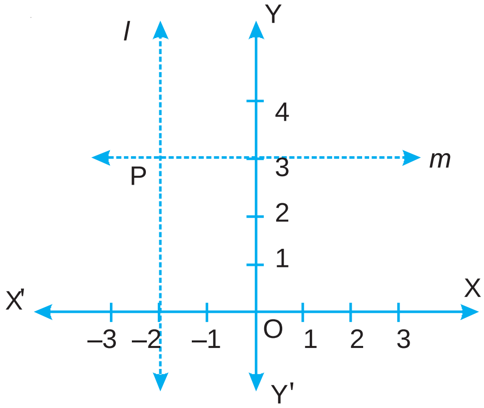

Fig 5.4

You will now learn to draw the graph of a linear equation in two variables. It should be noted that the graph of linear equation in two variables is a line and the coordinates of every point on the line satisfies the equation. If a point does not lie on the graph then its coordinates will not safisfy the equation. You also know that from two given points, one and only one line can be drawn. Therefore, it is sufficient to take any two points, i.e., values of the variables x and y which satisfy the equation. However, it is suggested that you should take three points to avoid any chance of a mistake occurring.

Example 5.15: Draw the graph of the equation 2x – 3y = 6.

Solution: Now choose values of x and y which satisfy the equation 2x – 3y = 6. It will be easy to write the equation by transforming it in any of the following form

$$2x = 3y + 6 \text{ or } 3y = 2x - 6$$

$$\Rightarrow \; x = \frac{3y+6}{2} \quad \text{or} \quad y = \frac{2x-6}{3}$$

Now by taking different values of x or y, you find the corresponding values of y or x. If we take different values of x in $y = \frac{2x-6}{3}$, we get corresponding values of y. If x = 0, we get y = – 2, x = 3 gives y = 0 and x = – 3 gives y = – 4.

You can represent these values in the following tabular form:

x	0	3	–3
y	– 2	0	– 4

The corresponding points in the plane are (0, –2), (3, 0) and (–3, –4). You can now plot these points and join them to get the line which represents the graph of the linear equation as shown here.

Note that all the three points must lie on the line.

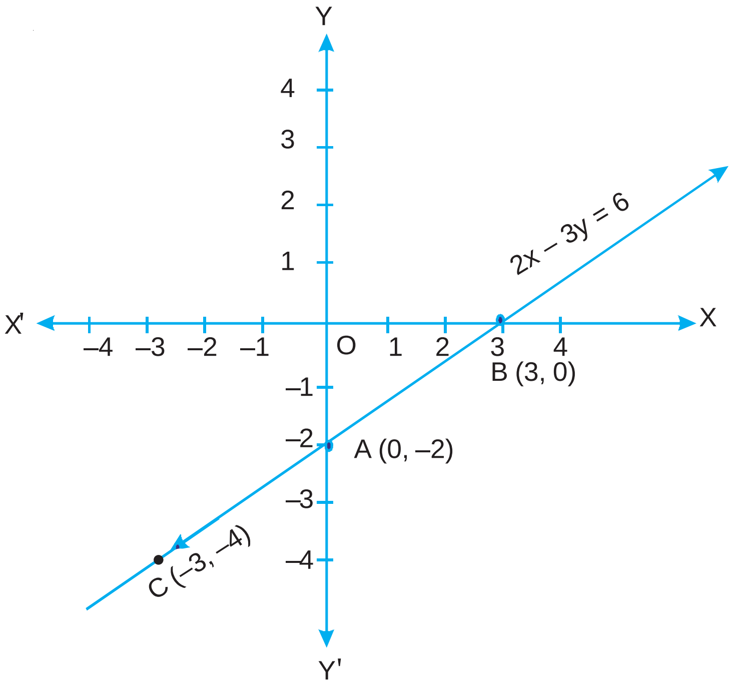

Fig 5.5

Example 5.16: Draw the graph of the equation x = 3.

Solution: It appears that it is a linear equation in one variable x. You can easily convert it into linear equation in two variables by writing it as

$$x + 0\,y = 3$$

Now you can have the following table for values of x and y.

x	3	3	3
y	3	0	1

Observe that for each value of y, the value of x is always 3. Thus, required points can be taken as (3, 3), (3, 0), (3, 1). The graph is shown in Fig. 5.6.

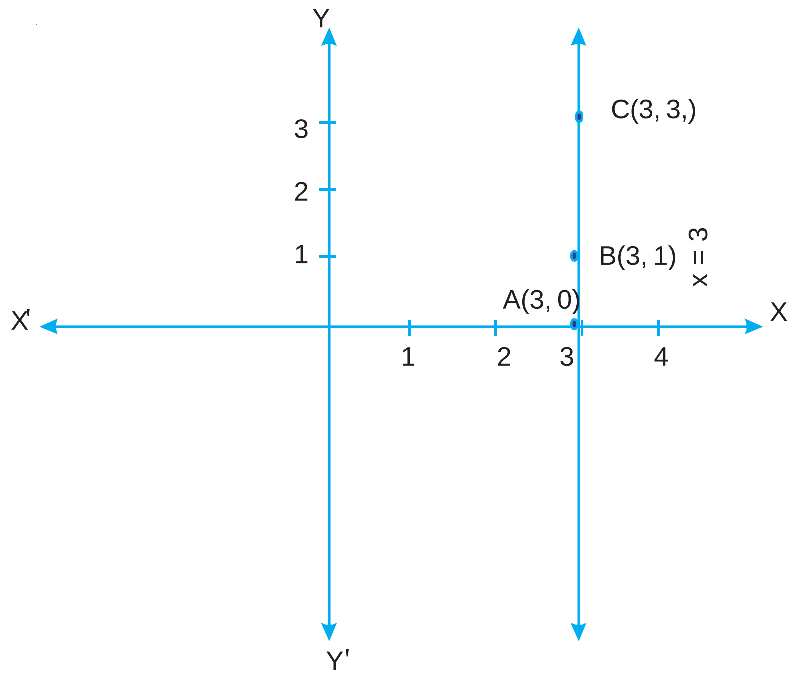

Fig. 5.6

CHECK YOUR PROGRESS 5.6

1. Plot the following points in the cartesian plane:

   (i) (3, 4) (ii) (–3, –2) (iii) (–2, 1)

   (iv) (2, –3) (v) (4, 0) (vi) (0, –3)

2. Draw the graph of each of the following linear equations in two variables:

   (i) $x + y = 5$ (ii) $3x + 2y = 6$

   (iii) $2x + y = 6$ (iv) $5x + 3y = 4$

5.7 SYSTEM OF LINEAR EQUATIONS IN TWO VARIABLES

Neha went to market and purchased 2 pencils and 3 pens for ₹ 19. Mary purchased 3 pencils and 2 pens for ₹ 16. What is the cost of 1 pencil and 1 pen? If the cost of one pencil is ₹ x and cost of one pen is ₹ y, then the linear equation in case of Neha is $2x + 3y = 19$ and for Mary it is $3x + 2y = 16$. To find the cost of 1 pencil and 1 pen, you have to find those values of x and y which satisfy both the equations, i.e.,

$2x + 3y = 19$

$3x + 2y = 16$

These two equations taken together are called system of linear equations in two variables and the values of x and y which satisfy both equations simultaneously is called the solution.

There are different methods for solving such equation. These are graphical method and algebraic method. You will first learn about graphical method and then algebraic method for solving such equations.

5.7.1 Graphical method

In this method, you have to draw the graphs of both linear equations on the same graph sheet. The graphs of the equations may be

(i) **Intersecting lines:** In this case, the point of intersection will be common solution of both simultaneous equations. The x-coordinate will give the value of x and y-coordinate will given value of y. In this case system will have a unique solution.

(ii) **Concident lines:** In this case each point on the common line will give the solution. Hence, system of equations will have infinitely many solutions.

(iii) **Parallel lines:** In this case, no point will be common to both equations. Hence, system of equations will have no solution.

Example 5.17: Solve the following system of equations:

$$x - 2y = 0 \qquad ...(1)$$

$$3x + 4y = 20 \qquad ...(2)$$

Solution: Let us draw the graphs of these equations. For this, you need atleast two solutions of each equation. We give these values in the following tables.

$x - 2y = 0$

x	0	2	–2
y	0	1	–1

$3x + 4y = 20$

x	0	4	6
y	5	2	1/2

Now plot these points on the same graph sheet as given below:

The two graphs intersect at the point P whose coordinates are (4, 2). Thus $x = 4$, $y = 2$ is the solution.

You can verify that $x = 4$, $y = 2$ satisfies both the equations.

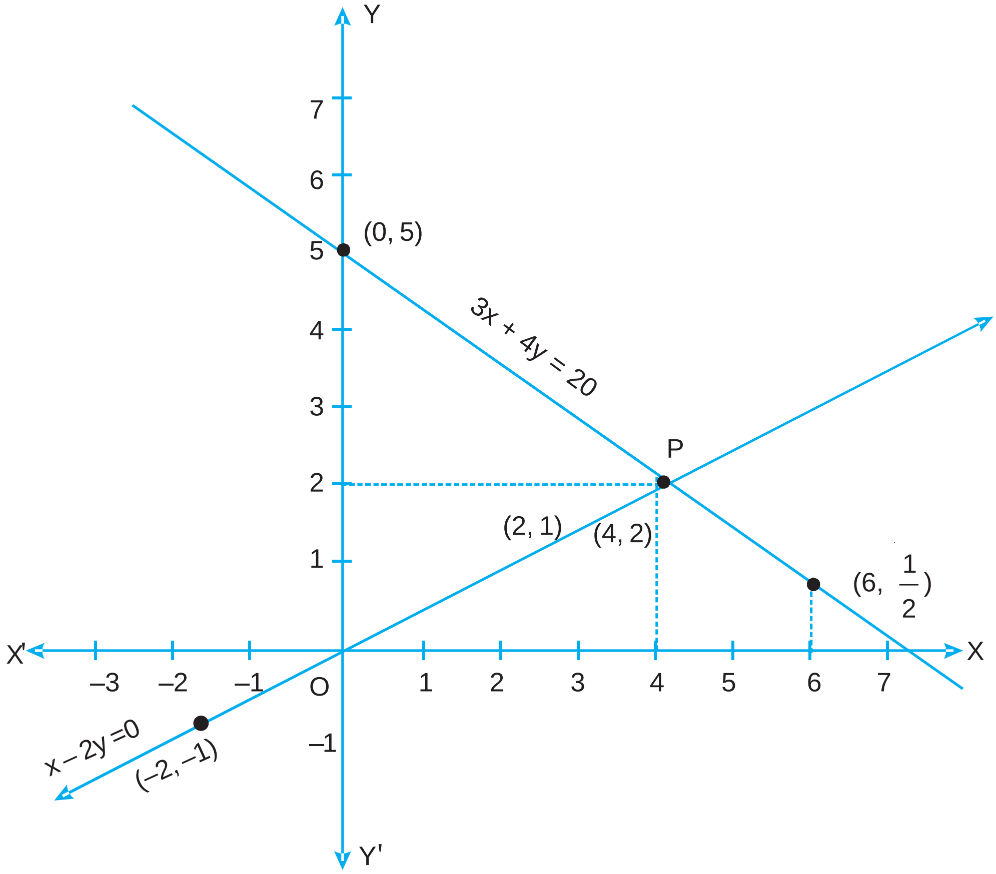

Fig. 5.7

Example 5.18: Solve the following system of equations:

$$x + y = 8 \qquad ...(1)$$

$$2x - y = 1 \qquad ...(2)$$

Solution: To draw the graph of these equation, make the following by selecting some solutions of each of the equation.

$x + y = 8$

x	3	4	5
y	5	4	3

$2x - y = 1$

x	0	1	2
y	–1	1	3

Now, plot the points (3, 5), (4, 4) and (5, 3) to get the graph of x + y = 8 and (0, –1), (1, 1) and (2, 3) to get the graph of 2x – y = 1 on the same graph sheet. The two lines intersect at the point P whose coordinates are (3, 5). Thus x = 3, y = 5, is the solution of the system of equations. You can verify that x = 3, y = 5 satisfies both equations simultaneously.

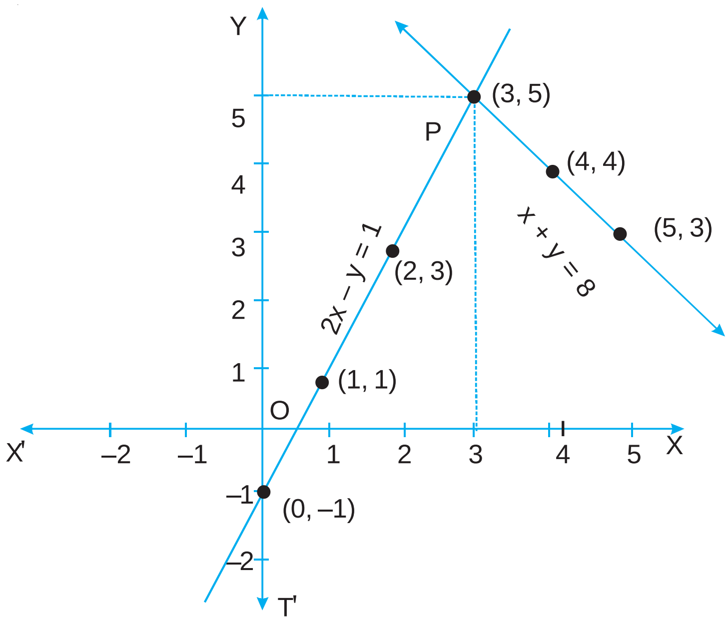

Fig. 5.8

Example 5.19: Solve the following system of equations:

$$x + y = 2 \qquad ...(1)$$

$$2x + 2y = 4 \qquad ...(2)$$

Solution: First make tables for some solutions of each of the equation.

$x + y = 2$

x	0	2	1
y	2	0	1

$2x + 2y = 4$

x	0	2	1
y	2	0	1

Now draw the graph of these equations by plotting the corresponding points.

You can see that graph of both the equations is the same. Hence, system of equations has infinitely many solutions. For example, $x = 0, y = 2$; $x = 1, y = 1$; $x = 2, y = 0$ etc. You can also observe that these two equations are essentially the same equation.

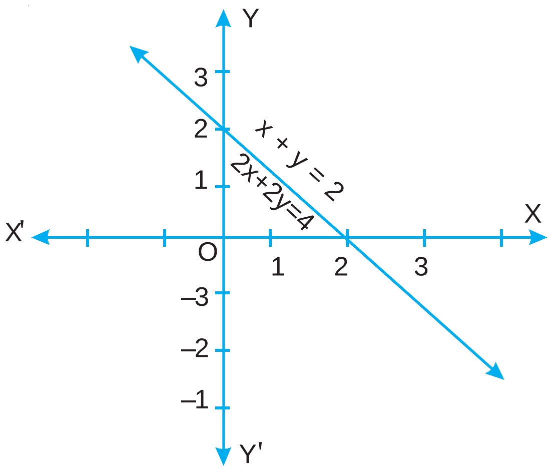

Fig. 5.9

Example 5.20: Solve the following system of equations:

$$2x - y = 4 \qquad ...(1)$$

$$4x - 2y = 6 \qquad ...(2)$$

Solution: Let us draw the graph of both equations by taking some solutions of each of the equation.

$2x - y = 4$

x	0	2	–1
y	–4	0	–6

$4x - 2y = 6$

x	0	1.5	2
y	–3	0	1

You can observe that these graphs are parallel lines. Since, they do not have any common point, the system of equations, therefore, has no solution.

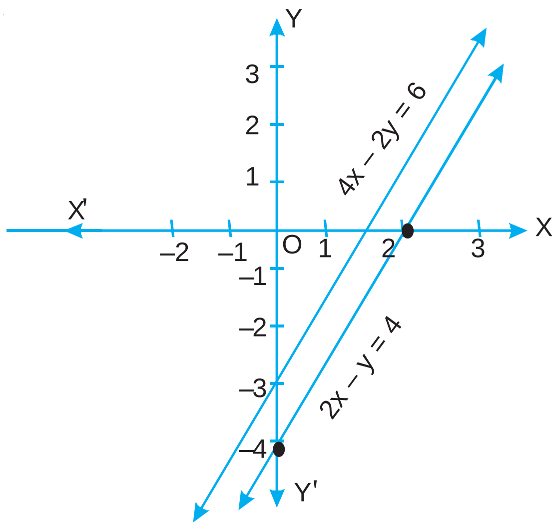

Fig. 5.10

CHECK YOUR PROGRESS 5.7

Solve the following system of equations graphically. Also, tell whether these have unique solution, infinitely many solutions or no solution.

1. $x - y = 3$
   $x + y = 5$
2. $2x + 3y = 1$
   $3x - y = 7$
3. $x + 2y = 6$
   $2x + 4y = 12$

4. $3x + 2y = 6$

   $6x + 4y = 18$

5. $2x + y = 5$

   $3x + 2y = 8$

5.7.2 Algebraic Method

There are several methods of solving system of two linear equations in two variables. You have learnt one method which is known as graphical method. We shall now discuss here two more methods, called algebraic methods. They are

(i) Substitution Method.

(ii) Elimination method.

Note: These methods are useful in case the system of equations has a unique solution.

Substitution Method: In this method, we find the value of one of the variable from one equation and substitute it in the second equation. This way, the second equation will be reduced to linear equaion in one variable which we have already solved. We explain this method through some examples.

Example 5.21: Solve the following system of equations by substitution method.

$$5x + 2y = 8 \qquad \text{...(1)}$$

$$3x - 5y = 11 \qquad \text{...(2)}$$

Solution: From (1), we get

$$2y = 8 - 5x$$

or $$y = \frac{1}{2}(8 - 5x) \qquad \text{...(3)}$$

Substituting the value of y in (2), we get

$$3x - \frac{5}{2}(8 - 5x) = 11$$

or $$6x - 5(8 - 5x) = 22 \qquad \text{[multiplying both sides by 2]}$$

or $$6x - 40 + 25x = 22$$

or $$31x = 40 + 22$$

or $$x = \frac{62}{31} = 2$$

Substituting the value of x = 2 in (3), we get

$$y = \frac{1}{2}(8 - 5\times 2) = \frac{1}{2}(8 - 10)$$

or $$y = -\frac{2}{2} = -1$$

So, the solution to the system of equations is x = 2, y = – 1.

Example 5.22: Solve the following system of equations by substitution method:

$$2x + 3y = 7 \qquad ...(1)$$

$$3x + y = 14 \qquad ...(2)$$

Solution: From equation (2), we get

$$y = 14 - 3x \qquad ...(3)$$

Substituting the value of y in (1), we get

$$2x + 3\,(14 - 3x) = 7$$

or $$2x + 42 - 9x = 7$$

or $$2x - 9x = 7 - 42$$

or $$-7x = -35$$

Therefore $$x = \frac{-35}{-7} = 5$$

Substituting the value of x in (3), we get

$$y = 14 - 3x = 14 - 3 \times 5$$

or $$y = 14 - 15 = -1$$

Hence, x = 5, y = – 1 is the solution.

Check: You can verify that x = 5, y = – 1 satisfies both the equations.

CHECK YOUR PROGRESS 5.8

Solve the following system of equations by substitution method:

1. $x + y = 14$
   $x - y = 2$

2. $2x + 3y = 11$
   $2x - 4y = -24$

3. $3x + 2y = 11$
   $2x + 3y = 4$

4. $7x - 2y = 1$
   $3x + 4y = 15$

Elimination Method: In this method, we eliminate one of the variable by multiplying both equations by suitable non-zero contants to make the coefficients of one of the variable numerically equal. Then we add or subtract one equation to or from the other so that one variable gets eliminated and we get an equation in one variable. We now consider some examples to illustrate this method.

Example 5.23: Solve the following system of equations using elimination method.

$$3x - 5y = 4 \qquad ...(1)$$

$$9x - 2y = 7 \qquad ...(2)$$

Solution: To elminate x, multiply equation (1) by 3 to make coefficient of x equal. You get the equations.

$$9x - 15y = 12 \qquad ...(3)$$

$$9x - 2y = 7 \qquad ...(4)$$

Subtracting (4) from (3), we get

$$9x - 15y - (9x - 2y) = 12 - 7$$

or $\quad 9x - 15y - 9x + 2y = 5$

or $\quad -13y = 5$

or $\quad y = -\dfrac{5}{13}$

Substituting $y = -\dfrac{5}{13}$ in equation (1), we get

$$3x - 5 \times \left(-\frac{5}{13}\right) = 4$$

or $\quad 3x + \dfrac{25}{13} = 4$

or $\quad 3x = 4 - \dfrac{25}{13} = \dfrac{27}{13}$

or $\quad x = \dfrac{9}{13}$

Therefore, $x = \frac{9}{13}$ and $y = -\frac{5}{13}$ is the required solution of the given system of equations.

Example 5.24: Solve the following system of equations using elimination method.

$$2x + 3y = 13 \qquad ...(1)$$

$$5x - 7y = -11 \qquad ...(2)$$

Solution: To eliminate y, multiply equation (1) by 7 and equation (2) by 3, we get

$$14x + 21y = 91 \qquad ...(3)$$

$$15x - 21y = -33 \qquad ...(4)$$

Adding (3) and (4), we get

$$29\,x = 58$$

or $$x = \frac{58}{29} = 2$$

Substituting x = 2 in (1), we get

$$2 \times 2 + 3y = 13$$

or $$3y = 13 - 4 = 9$$

or $$y = \frac{9}{3} = 3$$

Therefore, x = 2 and y = 3 is the solution of the given system of equations.

CHECK YOUR PROGRESS 5.9

Solve the following systems of equations by elimination method:

1. $3x + 4y = -6$
   $3x - y = 9$
2. $x + 2y = 5$
   $2x + 3y = 8$
3. $x - 2y = 7$
   $3x + y = 35$
4. $3x + 4y = 15$
   $7x - 2y = 1$
5. $2x + 3y = 4$
   $3x + 2y = 11$
6. $3x - 5y = 23$
   $2x - 4y = 16$

5.8 WORD PROBLEMS

Example 5.25: The perimeter of a rectangular garden is 20 m. If the length is 4 m more than the breadth, find the length and breadth of the garden.

Solution: Let the length of the garden be x m. Therefore, breadth of garden = $(x - 4)$ m. Since, perimeter is 20 m, so

$$2\,[x + (x - 4)] = 20$$

or $\quad 2(2x - 4) = 20$

or $\quad 2x - 4 = 10$

or $\quad 2x = 10 + 4 = 14$

or $\quad x = 7$

Hence, length = 7 m and breadth = $7 - 4 = 3$ m.

Alternatively, you can solve the problem using two variables. Proceed as follows:

Let the length of garden = x m

and $\quad$ width of garden = y m

Therefore $\quad x = y + 4 \qquad ...(1)$

Also, perimeter is 20 m, therefore

$$2(x + y) = 20$$

or $\quad x + y = 10 \qquad ...(2)$

Solving (1) and (2), we get $x = 7$, $y = 3$

Hence, length = 7 m and breadth = 3m

Example 5.26: Asha is five years older than Robert. Five years ago, Asha was twice as old as Robert was then. Find their present ages.

Solution: Let present age of Asha be x years

and $\quad$ present age of Robert be y years

Therefore, $\quad x = y + 5$

or $\quad x - y = 5 \qquad ...(1)$

5 years ago, Asha was $x - 5$ years and Robert was $(y - 5)$ years old.

Therefore, $\quad x - 5 = 2(y - 5)$

or $\quad x - 2y = -5 \qquad ...(2)$

Solving (1) and (2), we get $y = 10$ and $x = 15$

Hence, present age of Asha = 15 years and present age of Robert = 10 years.

Example 5.27: Two places A and B are 100 km apart. One car starts from A and another from B at the same time. If they travel in the same direction, they meet after 5 hours. If they travel towards each other, they meet in 1 hour. What are the speeds of the two cars. Assume that the speed of car at A is more than the speed of car at B.

Solution: Let speed of the car starting from A be x km/h

and speed of the car starting from B be y km/h

Therefore, the distance travelled by car at A in 5 hours = 5x km

and the distance travelled by car at B in 5 hours = 5y km

Since they meet after 5 hours when they travel in the same direction, the car at A has travelled 100 km more than the car at B. Therefore,

$$5x - 5y = 100$$

or $$x - y = 20 \qquad ...(1)$$

When they travel towards each other, they meet after 1 hour. It means, total distance travelled by car at A and car at B in 1 hour is 100 km

Therefore $$x + y = 100 \qquad ...(2)$$

Solving (1) and (2), we get x = 60 and y = 40

Therefore, the speed of car at A = 60 km/h and

the speed of car at B = 40 km/h.

CHECK YOUR PROGRESS 5.10

1. Rahim's father is three times as old as Rahim. If sum of their ages is 56 years, find their ages.
2. Rita has 18m of cloth. She cut it into two pieces in such a way that one piece is 4 m longer than the other. What is the length of shorter piece.
3. A total of Rs 50000 is to be distributed among 200 persons as prizes. A prize is either Rs 500 or Rs 100. Find the number of each type of prizes.
4. A purse contain Rs 2500 in notes of denominations of 100 and 50. If the number of 100 rupee notes is one more than that of 50 rupee notes, find the number of notes of each denomination.

LET US SUM UP

- An equation in one variable of degree one is called a linear equation in variable.
- The general form of a linear equation in one variable is $ax + b = 0$, $a \neq 0$, a and b are real numbers.
- The value of the variable which satisfies the linear equation is called its solution or root.
- To solve a word problem, it is first translated into algebriac statements and then solved.
- The general form of a linear equation in two variables is $ax + by + c = 0$, where a, b, c are real numbers and atleast one of a or b is non zero.
- The equation $ax + c = 0$ can be expressed as linear equation in two variables as $ax + 0y + c = 0$.
- To draw the graph of a linear equation in two variables, we find atleast two points in plane whose coordinates are solutions of the equation and plot them.
- The graph of a linear equaion in two variables is a line.
- To solve two simultaneous equations in two variables, we draw their graphs on the same graph paper.

  (i) if graph is intersecting lines, point of intersection gives unique solution.

  (ii) If graph is the same line, system has infinitely many solutions

  (iii) If graph is parallel lines, system of equation has no solution
- Algebraic methods of solving system of linear equations are

  (i) Substitution method

  (ii) Elimination method
- To solve word problems, we tranlate the given information (data) into linear equations and solve them.

TERMINAL EXERCISE

1. Choose the correct option:

   (i) Which one of the following is a linear equation in one variable?

   (A) $2x + 1 = y - 3$ (B) $3t - 1 = 2t + 5$

   (C) $2x - 1 = x^2$ (D) $x^2 - x + 1 = 0$

   (ii) Which one of the following is not a linear equation?

   (A) $5 + 4x = y + 3$ (B) $x + 2y = y - x$

(C) $3 - x = y^2 + 4$ (D) $x + y = 0$

(iii) Which of the following numbers is the solution of the equation $2(x + 3) = 18$?

(A) 6 (B) 12

(C) 13 (D) 21

(iv) The value of x, for which the equation $2x - (4 - x) = 5 - x$ is satisfied, is:

(A) 4.5 (B) 3

(C) 2.25 (D) 0.5

(v) The equation $x - 4y = 5$ has

(A) no solution (B) unique solution

(C) two solutions (D) infinitely many solutions

2. Solve each of the following equations

(i) $2z + 5 = 15$ (ii) $\frac{x+2}{3} = -2$

(iii) $\frac{4-2y}{3} + \frac{y+1}{2} = 1$ (iv) $2.5x - 3 = 0.5x + 1$

3. A certain number increased by 8 equals 26. Find the number.

4. Present ages of Reena and Meena are in the ration 4 : 5. After 8 years, the ratio of their ages will be 5 : 6. Find their present ages.

5. The denominator of a rational number is greater than its numerator by 8. If the denominaor is decreased by 1 and numerator is increased by 17, the number obtained is $\frac{3}{2}$. Find the rational number

6. Solve the following system of equations graphically:

(i) $x - 2y = 7$, $x + y = -2$ (ii) $4x + 3y = 24$, $3y - 2x = 6$

(iii) $x + 3y = 6$, $2x - y = 5$ (iv) $2x - y = 1$, $x + y = 8$

7. Solve the following system of equations :

(i) $x + 2y - 3 = 0$, $x - 2y + 1 = 0$ (ii) $2x + 3y = 3$, $3x + 2y = 2$

(iii) $3x - y = 7$, $4x - 5y = 2$ (iv) $5x - 2y = -7$, $2x + 3y = -18$

8. The sum of the digits of a two-digit number is 11. If the digits are reversed, the new number is 27 less than the original number. Find the original number.

9. Three years ago Atul's age was four times Parul's age. After 5 years from now, Atul's age will be two times Parul's age. Find their present ages.

10. The perimeter of a rectangular plot of land is 32 m. If the length is increased by 2m and breadth is decreased by 1 m, the area of the plot remains the same. Find the length and breadth of the plot.

ANSWERS TO CHECK YOUR PROGRESS

5.1

1. (i)   2. (i)

5.2

1. $15 - 2x = 7$
2. $0.1x = 10$
3. $6y = 96$
4. $t + 15 = 4t$

5.3

1. $x = 13$ 2. $y = 12$ 3. $z = 0$

4. $y = 9$ 5. $x = 5$

5.4

1. 39, 46
2. 15 years, 50 years
3. 22 cm, 11 cm
4. 25

5.5

1. (i) $2(x + y) = 98$

   (ii) $y = 2x + 10$, where age of son = x years, age of father = y years

   (iii) $x + 10 = y$

   (iv) $2x + 3y = 120$

2. True   3. False

5.7

1. $x = 4, y = 1$, unique solution
2. $x = 2, y = -1$, unique solution
3. Infinitely many solutions
4. No solution
5. $x = 2, y = 1$, unique solution

5.8

1. $x = 8, y = 6$ 2. $x = -2, y = 5$
3. $x = 5, y = -2$ 4. $x = 1, y = 3$

5.9

1. $x = 2, y = -3$ 2. $x = 1, y = 2$
3. $x = 11, y = 2$ 4. $x = 1, y = 3$
5. $x = 5, y = -2$ 6. $x = 6, y = -1$

5.10

1. 14 years, 42 years
2. 7 m
3. 75 prizes Rs 500 and 125 prizes of Rs 100 each.
4. 17 of Rs 100 each and 16 of Rs 50 each.

ANSWERS TO TERMINAL EXERCISE

1. (i) (B) (ii) (C) (iii) (A) (iv) (C) (v) (D)
2. (i) $z = 5$ (ii) $x = -8$ (iii) $y = 5$ (iv) $x = 2$
3. 18
4. Age of Reena = 32 years, age of Meena = 40 years
5. $\frac{13}{21}$

6. (i) $x = 1, y = -3$ (ii) $x = 3, y = 4$

   (iii) $x = 3, y = 1$ (iv) $x = 3, y = 5$

7. (i) $x = 1, y = 1$ (ii) $x = 0, y = 1$

   (iii) $x = 3, y = 2$ (iv) $x = -3, y = -4$

8. 74

9. Atul: 19 years, Parul: 7 years

10. 10 m, 6m

21len06

6

QUADRATIC EQUATIONS

In this lesson, you will study about quadratic equations. You will learn to identify quadratic equations from a collection of given equations and write them in standard form. You will also learn to solve quadratic equations and translate and solve word problems using quadratic equations.

OBJECTIVES

After studying this lesson, you will be able to

- *identify a quadratic equation from a given collection of equations;*
- *write quadratic equations in standard form;*
- *solve quadratic equations by (i) factorization and (ii) using the quadratic formula;*
- *solve word problems using quadratic equations.*

EXPECTED BACKGROUND KNOWLEDGE

- Polynomials
- Zeroes of a polynomial
- Linear equations and their solutions
- Factorisation of a polynomial

6.1 QUADRATIC EQUATIONS

You are already familiar with a polynomial of degree two. A polynomial of degree two is called a quadratic polynomial. When a quadratic polynomial is equated to zero, it is called **a quadratic equation**. In this lesson, you will learn about quadratic equations in one variable only. Let us consider some examples to identify a quadratic equation from a collection of equations

Example 6.1: Which of the following equations are quadratic equations?

(i) $3x^2 = 5$ (ii) $x^2 + 2x + 3 = 0$

(iii) $x^3 + 1 = 3x^2$ (iv) $(x + 1)(x + 3) = 2x + 1$

(v) $x + \frac{1}{x} = \frac{5}{2}$ (v) $x^2 + \sqrt{x} + 1 = 0$

Solution:

(i) It is a quadratic equation since $3x^2 = 5$ can be written as $3x^2 - 5 = 0$ and $3x^2 - 5$ is a quadratic polynomial.

(ii) $x^2 + 2x + 3 = 0$ is a quadratic equation as $x^2 + 2x + 3$, is a polynomial of degree 2.

(iii) $x^3 + 1 = 3x^2$ can be written as $x^3 - 3x^2 + 1 = 0$. LHS is not a quadratic polynomial since highest exponent of x is 3. **So, the equation is not a quadratic equation.**

(iv) $(x + 1)(x + 3) = 2x + 1$ is a quadratic equation, since $(x + 1)(x + 3) = 2x + 1$ can be written as

$x^2 + 4x + 3 = 2x + 1$

or $x^2 + 2x + 2 = 0$

Now, LHS is a polynomial of degree 2, hence $(x + 1)(x + 3) = 2x + 1$ is a quadratic equation.

(v) $x + \frac{1}{x} = \frac{5}{2}$ is not a quadratic equation.

However, it can be reduced to quadratic equation as shown below:

$$x + \frac{1}{x} = \frac{5}{2}$$

or $\frac{x^2 + 1}{x} = \frac{5}{2}, x \neq 0$

or $2(x^2 + 1) = 5x, x \neq 0$

or $2x^2 - 5x + 2 = 0, x \neq 0$

(vi) $x^2 + \sqrt{x} + 1 = 0$ is not a quadratic equation as $x^2 + \sqrt{x} + 1$ is not a quadratic polynomial (Why?)

CHECK YOUR PROGRESS 6.1

1. Which of the following equations are quadratic equations?

   (i) $3x^2 + 5 = x^3 + x$ (ii) $\sqrt{3}\,x^2 + 5x + 2 = 0$

   (iii) $(5y + 1)(3y - 1) = y + 1$ (iv) $\dfrac{x^2+1}{x+1} = \dfrac{5}{2}$

   (v) $3x + 2x^2 = 5x - 4$

6.2 STANDARD FORM OF A QUADRATIC EQUATION

A quadratic equation of the form $ax^2 + bx + c = 0$, $a > 0$ where a, b, c, are constants and x is a variable is called a quadratic equation in the standard form. Every quadratic equation can always be written in the standard form.

Example 6.2: Which of the following quadratic equations are in standard form? Those which are not in standard form, express them in standard form.

(i) $2 + 3x + 5x^2 = 0$ (ii) $3x^2 - 5x + 2 = 0$

(iii) $7y^2 - 5y = 2y + 3$ (iv) $(z + 1)(z + 2) = 3z + 1$

Solution: (i) It is not in the standard form. Its standard form is $5x^2 + 3x + 2 = 0$

(ii) It is in standard form

(iii) It is not in the standard form. It can be written as

$7y^2 - 5y = 2y + 3$

or $7y^2 - 5y - 2y - 3 = 0$

or $7y^2 - 7y - 3 = 0$

which is now in the standard form.

(iv) It is not standard form. It can be rewritten as

$(z + 1)(z + 2) = 3z + 1$

or $z^2 + 3z + 2 = 3z + 1$

or $z^2 + 3z - 3z + 2 - 1 = 0$

or $z^2 + 1 = 0$

or $z^2 + 0z + 1 = 0$

which is now in the standard form.

CHECK YOUR PROGRESS 6.2

1. Which of the following quadratic equations are in standard form? Those, which are not in standard form, rewrite them in standard form:

   (i) $3y^2 - 2 = y + 1$ (ii) $5 - 3x - 2x^2 = 0$

   (iii) $(3t - 1)(3t + 1) = 0$ (iv) $5 - x = 3x^2$

6.3 SOLUTION OF A QUADRATIC EQUATION

You have learnt about the zeroes of a polynomial. A zero of a polynomial is that real number, which when substituted for the variable makes the value of the polynomial zero. In case of a quadratic equation, the value of the variable for which LHS and RHS of the equation become equal is called a root or solution of the quadratic equation. You have also learnt that if α is a zero of a polynomial p(x), then $(x - \alpha)$ is a factor fo p(x) and conversely, if $(x - \alpha)$ is a factor of a polynomial, then α is a zero of the polynomial. You will use these results in finding the solution of a quadratic equation. There are two algebraic methods for finding the solution of a quadratic equation. These are (i) Factor Method and (ii) Using the Quadratic Formula.

Factor Method

Let us now learn to find the solutions of a quadratic equation by factorizing it into linear factors. The method is illustrated through examples.

Example 6.3: Solve the equation $(x - 4)(x + 3) = 0$

Solution: Since, $(x - 4)(x + 3) = 0$, therefore,

either $x - 4 = 0$, or $x + 3 = 0$

or $x = 4$ or $x = -3$

Therefore, $x = 4$ and $x = -3$ are solutions of the equation.

Example 6.4: Solve the equation $6x^2 + 7x - 3 = 0$ by factorisation.

Solution: Given $6x^2 + 7x - 3 = 0$

By breaking middle term, we get

$6x^2 + 9x - 2x - 3 = 0$ [since, $6 \times (-3) = -18$ and $-18 = 9 \times (-2)$ and $9 - 2 = 7$]

or $3x(2x + 3) - 1(2x + 3) = 0$

or $(2x + 3)(3x - 1) = 0$

This gives $2x + 3 = 0$ or $3x - 1 = 0$

or $x = -\dfrac{3}{2}$ or $x = \dfrac{1}{3}$

Therefore, $x = -\frac{3}{2}$ and $x = \frac{1}{3}$ are solutions of the given equation.

Example 6.5: Solve $x^2 + 2x + 1 = 0$

Solution: We have $x^2 + 2x + 1 = 0$

or $(x + 1)^2 = 0$

or $x + 1 = 0$

which gives $x = -1$

Therefore, $x = -1$ is the only solution.

Note: In Examples 6.3 and 6.4, you saw that equations had two distinct solutions. However, in Example 6.5, you got only one solution. We say that it has two solutions and these are coincident.

CHECK YOUR PROGRESS 6.3

1. Solve the following equations using factor method.

(i) $(2x + 3)(x + 2) = 0$ (ii) $x^2 + 3x - 18 = 0$

(iii) $3x^2 - 4x - 7 = 0$ (iv) $x^2 - 5x - 6 = 0$

(v) $25x^2 - 10x + 1 = 0$ (vi) $4x^2 - 8x + 3 = 0$

Quadratic Formula

Now you will learn to find a formula to find the solution of a quadratic equation. For this, we will rewrite the general quadratic equation $ax^2 + bx + c = 0$ by completing the square.

We have $ax^2 + bx + c = 0$

Multiplying both sides by '4a' to make the coefficient of x^2 a perfect square, of an even number, we get

$$4a^2x^2 + 4abx + 4ac = 0$$

or $(2ax)^2 + 2(2ax)b + (b)^2 + 4ac = b^2$ [adding b^2 to both sides]

or $(2ax)^2 + 2(2ax)b + (b)^2 = b^2 - 4ac$

or $(2ax + b)^2 = \left\{\pm\sqrt{b^2 - 4ac}\right\}^2$

or $2ax + b = \pm\sqrt{b^2 - 4ac}$

or $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This gives two solutions of the quadratic equation $ax^2 + bx + c = 0$. The solutions (roots) are:

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Here, the expression $(b^2 - 4ac)$, denoted by D, is called **Discriminant**, because it determines the number of solutions or nature of roots of a quadratic equation.

For a quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$, if

(i) $D = b^2 - 4ac > 0$, the equation has two real distinct roots, which are $\frac{-b + \sqrt{b^2 - 4ac}}{2a}$

and $\frac{-b - \sqrt{b^2 - 4ac}}{2a}$.

(ii) $D = b^2 - 4ac = 0$, then equation has two real equal roots, each equal to $\frac{-b}{2a}$

(iii) $D = b^2 - 4ac < 0$, the equation will not have any real root, since square root of a negative real number is not a real number.

Thus, a quadratic equation will have at the most two roots.

Example 6.6: Without determining the roots, comment on the nature (number of solutions) of roots of the following equations:

(i) $3x^2 - 5x - 2 = 0$

(ii) $2x^2 + x + 1 = 0$

(iii) $x^2 + 2x + 1 = 0$

Solution: (i) The given equation is $3x^2 - 5x - 2 = 0$. Comparing it with $ax^2 + bx + c = 0$, we get $a = 3$, $b = -5$ and $c = -2$.

Now $D = b^2 - 4ac \quad = (-5)^2 - 4 \times 3 \times (-2)$

$= 25 + 24 = 49$

Since, $D > 0$, the equation has two real distinct roots.

(ii) Comparing the equation $2x^2 + x + 1 = 0$ with $ax^2 + bx + c = 0$,

we get $a = 2$, $b = 1$, $c = 1$

Now $D = b^2 - 4ac = (1)^2 - 4 \times 2 \times 1 = 1 - 8 = -7$

Since, $D = b^2 - 4ac < 0$, the equation does not have any real root.

(iii) Comparing the equation $x^2 + 2x + 1 = 0$ with $ax^2 + bx + c = 0$,

we get $a = 1, b = 2, c = 1$

Now $D = b^2 - 4ac = (2)^2 - 4 \times 1 \times 1 = 0$

Since, $D = 0$, the equation has two equal roots.

Example 6.7: Using quadratic formula, find the roots of the equation $6x^2 - 19x + 15 = 0$

Solution: Comparing the given equation with $ax^2 + bx + c = 0$

We get, $a = 6, b = -19, c = 15$

Now $D = b^2 - 4ac = (-19)^2 - 4 \times 6 \times 15$

$= 361 - 360 = 1$

Therefore, roots are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{19 \pm \sqrt{1}}{12} = \frac{19 \pm 1}{12}$$

So, roots are $\frac{19+1}{12} = \frac{5}{3}$ and $\frac{19-1}{12} = \frac{3}{2}$

Thus, the two roots are $\frac{5}{3}$ and $\frac{3}{2}$.

Example 6.8: Find the value of m sothat the equation $3x^2 + mx - 5 = 0$ has equal roots.

Solution: Comparing the given equation with $ax^2 + bx + c = 0$

We have, $a = 3, b = m, c = -5$

For equal roots

$D = b^2 - 4ac = 0$

or $m^2 - 4 \times 3 \times (-5) = 0$

or $m^2 = 60$

This gives $m = \pm 2\sqrt{15}$

Hence, for $m = \pm 2\sqrt{15}$, the equation will have equal roots.

CHECK YOUR PROGRESS 6.4

1. Without determining the roots, comment on nature of roots of following equations:

   (i) $3x^2 - 7x + 2 = 0$ (ii) $4x^2 - 12x + 9 = 0$ (iii) $25x^2 + 20x + 4 = 0$

   (iv) $x^2 - x + 1$

2. Solve the following equations using quadratic formula:

   (i) $y^2 - 14y - 12 = 0$ (ii) $x^2 - 5x = 0$ (iii) $x^2 - 15x + 50 = 0$

3. Find the value of m so that the following equations have equal roots:

   (i) $2x^2 - mx + 1 = 0$ (ii) $mx^2 + 3x - 5 = 0$

   (iii) $3x^2 - 6x + m = 0$ (iv) $2x^2 + mx - 1 = 0$

6.4 WORD PROBLEMS

We will now solve some problems which involve the use of quadratic equations.

Example 6.9: The sum of squares of two consecutive odd natural numbers is 74. Find the numbers.

Solution: Let two consecutive odd natural numbers be x and x + 2. Since, sum of their squares is 74. we have

$$x^2 + (x + 2)^2 = 74$$

or $$x^2 + x^2 + 4x + 4 = 74$$

or $$2x^2 + 4x - 70 = 0$$

or $$x^2 + 2x - 35 = 0$$

or $$x^2 + 7x - 5x - 35 = 0$$

or $$x(x + 7) - 5(x + 7) = 0$$

or $$(x + 7)(x - 5) = 0$$

Therefore $x + 7 = 0$ or $x - 5 = 0$

or $$x = -7 \text{ or } x = 5$$

Now, x can not be negative as it is a natural number. Hence $x = 5$

So, the numbers are 5 and 7.

Example 6.10: The sum of the areas of two square fields is 468 m^2. If the difference of their perimeter is 24 m, find the sides of the two squares.

Solution: Let the sides of the bigger square be x and that of the smaller square be y.

Hence, perimeter of bigger square = 4x

and perimeter of smaller square = 4y

Therefore, $\quad 4x - 4y = 24$

or $\quad x - y = 6$

or $\quad x = y + 6 \quad$ ....(1)

Also, since sum of areas of two squares is 468 m^2

Therefore, $\quad x^2 + y^2 = 468 \quad$ ....(2)

Substituting value of x from (1) into (2), we get

$$(y + 6)^2 + y^2 = 468$$

or $\quad y^2 + 12y + 36 + y^2 = 468$

or $\quad 2y^2 + 12y - 432 = 0$

or $\quad y^2 + 6y - 216 = 0$

Therefore $\quad y = \dfrac{-6 \pm \sqrt{36+864}}{2} = \dfrac{-6 \pm \sqrt{900}}{2}$

or $\quad y = \dfrac{-6 \pm 30}{2}$

Therefore, $\quad y = \dfrac{-6+30}{2}$ or $\dfrac{-6-30}{2}$

or $\quad y = 12$ or -18

Since, side of square can not be negative, so y = 12

Therefore, $\quad x = y + 6 = 12 + 6 = 18$

Hence, sides of squares are 18 m and 12 m.

Example 6.11: The product of digits of a two digit number is 12. When 9 is added to the number, the digits interchange their places. Determine the number.

Solution: Let the digit at ten's place be x

and digit at unit's place be y

Therefore, number = $10x + y$

When digits are interchanged, the number becomes $10y + x$

Therefore $\quad 10x + y + 9 = 10y + x$

or $\quad 10x - x + y - 10y = -9$

or $\quad 9x - 9y = -9$

or $\quad x - y = -1$

or $\quad x = y - 1 \qquad \text{...(1)}$

Also, product of digits is 12

Hence, $\quad xy = 12 \qquad \text{...(2)}$

Substituting value of x from (1) into (2), we get

$$(y - 1)y = 12$$

or $\quad y^2 - y - 12 = 0$

or $\quad (y - 4)(y + 3) = 0$

Hence, $\quad y = 4$ or $y = -3$

Since, digit can not be negative, $y = 4$

Hence $\; x = y - 1 = 4 - 1 = 3$

Therefore, the number is 34.

Example 6.12: The sum of two natural numbers is 12. If sum of their reciprocals is $\frac{4}{9}$, find the numbers.

Solution: Let one number be x

Therefore, other number $= 12 - x$

Since, sum of their reciprocals is $\frac{4}{9}$, we get

$$\frac{1}{x} + \frac{1}{12 - x} = \frac{4}{9}, \quad x \neq 0, \; 12 - x \neq 0$$

or $$\frac{12 - x + x}{x(12 - x)} = \frac{4}{9}$$

or $$\frac{12}{12x - x^2} = \frac{4}{9}$$

or $$\frac{12 \times 9}{4} = 12x - x^2$$

or $\quad 27 = 12x - x^2$

or $\quad x^2 - 12x + 27 = 0$

or $\quad (x-3)(x-9) = 0$

It gives $x = 3$ or $x = 9$

When first number x is 3, other number is $12 - 3 = 9$ and when first number x is 9, other number is $12 - 9 = 3$.

Therefore, the required numbers are 3 and 9.

CHECK YOUR PROGRESS 6.5

1. The sum of the squares of two consecutive even natural numbers is 164. Find the numbers.
2. The length of a rectangular garden is 7 m more than its breadth. If area of the garden is 144 m^2, find the length and breadth of the garden.
3. The sum of digits of a two digit number is 13. If sum of their squares is 89, find the number.
4. The digit at ten's place of a two digit number is 2 more than twice the digit at unit's place. If product of digits is 24, find the two digit number.
5. The sum of two numbers is 15. If sum of their reciprocals is $\frac{3}{10}$, find the two numbers.

LET US SUM UP

- An equation of the form $ax^2 + bx + c = 0$, $a \neq 0$ and a, b, c are real numbers is called a quadratic equation in standard form.
- The value(s) of the variable which satisfy a quadratic equation are called it roots or solutions.
- The zeros of a quadratic polynomial are the roots or solutions of the corresponding quadratic equation.
- If you can factorise $ax^2 + bx + c = 0$, $a \neq 0$, into product of linear factors, then the roots of the quadratic equation $ax^2 + bx + c = 0$, can be obtained by equating each factor to zero.
- Roots of the quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$ are given by

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- $b^2 - 4ac$ is called discriminant of the quadratic equation. $ax^2 + bx + c = 0, a \neq 0$ It is usually denoted by D.

  (i) If $D > 0$, then the quadratic equation has two real unequal (distinct) roots.

  (ii) If $D = 0$, then the quadratic equation has two equal (coincident) roots.

  (iii) If $D < 0$, then the quadratic equation has no real root.

TERMINAL EXERCISE

1. Which of the following are quadratic equations?

   (i) $y\left(\sqrt{5}y - 3\right) = 0$ (ii) $5x^2 - 3\sqrt{x} + 8 = 0$

   (iii) $3x - \frac{1}{x} = 5$ (iv) $x(2x + 5) = x^2 + 5x + 7$

2. Solve the following equations by factorisation method:

   (i) $(x - 8)(x + 4) = 13$ (ii) $3y^2 - 7y = 0$

   (iii) $x^2 + 3x - 18 = 0$ (iv) $6x^2 + x - 15 = 0$

3. Find the value of m for which $5x^2 - 3x + m = 0$ has equal roots.
4. Find the value of m for which $x^2 - mx - 1 = 0$ has equal roots.
5. Solve the following quadratic equations using quadratic formula:

   (i) $6x^2 - 19x + 15 = 0$ (ii) $x^2 + x - 1 = 0$

   (iii) $21 + x = 2x^2$ (iv) $2x^2 - x - 6 = 0$

6. The sides of a right angled triangle are $x - 1$, x and $x + 1$. Find the value of x and hence the sides of the triangle.
7. the sum of squares of two consecutive odd integers is 290. Find the integers.
8. The hypotenuse of a right angled triangle is 13 cm. If the difference of remaining two sides is 7 cm, find the remaining two sides.
9. The sum of the areas of two squares is 41 cm^2. If the sum of their perimeters is 36 cm, find the sides of the two squares.
10. A right angled isosceles triangle is inscribed in a circle of radius 5 cm. Find the sides of the triangle.

ANSWERS TO CHECK YOUR PROGRESS

6.1

1. (ii), (iii), (v)

6.2

1. (i) No, $3y^2 - y - 3 = 0$ (ii) No, $2x^2 + 2x - 5 = 0$

   (iii) No, $6t^2 + t - 1 = 0$ (iv) No, $3x^2 + x - 5 = 0$

6.3

1. (i) $\frac{3}{2}, -2$ (ii) 3, – 6 (iii) $\frac{7}{3}, -1$

   (iv) 2, 3 (v) $\frac{1}{5}, \frac{1}{5}$ (vi) $\frac{3}{2}, \frac{1}{2}$

6.4

1. (i) Two real distinct roots

   (ii) Two real equal roots

   (iii) Two real equal roots

   (iv) No real roots

2. (i) $7 \pm \sqrt{37}$ (ii) 0, 5 (iii) 5, 10

3. (i) $\pm 2\sqrt{2}$ (ii) $\frac{9}{20}$ (iii) 3 (iv) For no value of m

6.5

1. 8, 10 2. 16m, 9m 3. 85, 58

4. 83 (v) 5, 10

ANSWERS TO TERMINAL EXERCISE

1. (i), (iv)

2. (i) 8, 4 (ii) 0, $\frac{7}{3}$ (iii) 3, – 6 (iv) $\frac{3}{2}, -\frac{5}{3}$

3. $\frac{9}{20}$

4. For no value of m

5. (i) $\frac{3}{2}, \frac{5}{3}$ (ii) $\frac{-1 \pm \sqrt{5}}{2}$ (iii) $\frac{7}{2}, -3$ (iv) $2, \frac{3}{2}$

6. 3, 4, 5

7. 11, 13 or –13 , –11

8. 5 cm, 12 cm

9. 5 cm, 4 cm

10. $5\sqrt{2}$ cm, $5\sqrt{2}$ cm, 10 cm

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7

ARITHMETIC PROGRESSIONS

In your daily life you must have observed that in nature, many things follow patterns such as petals of flowers, the holes of a honey-comb, the spirals on a pine apple etc. In this lesson, you will study one special type of number pattern called Arithmetic Progression (AP). You will also learn to find general term and the sum of first n terms of an arithmetic progression.

OBJECTIVES

After studying this lesson, you will be able to

- *identify arithmetic progression from a given list of numbers;*
- *determine the general term of an arithmetic progression;*
- *find the sum of first n terms of an arithmetic progression.*

PREVIOUS BACKGROUND KNOWLEDGE

- Knowledge of number system
- Operations on number system

7.1 SOME NUMBER PATTERNS

Let us consider some examples:

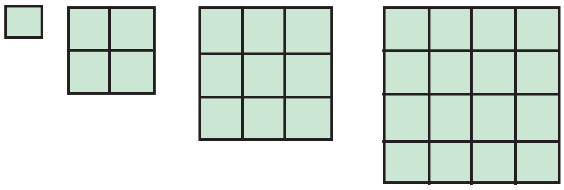
(i) Rita deposits ₹ 1000 in a bank at the simple interest of 10% per annum. The amount at the end of first, second, third and fourth years, in rupees will be respectively

1100, 1200, 1300, 1400

Do you observe any pattern? You can see that amount increases every year by a fixed amount of ₹ 100.

(ii) The number of unit squares in a square with sides 1, 2, 3, 4, ... units are respectively

1, 4, 9, 16, ...



Can you see any pattern in the list of these numbers? You can observe that

$1 = 1^2,\ 4 = 2^2,\ 9 = 3^2,\ 16 = 4^2, \ldots$

i.e., these are squares of natural numbers.

Now consider some more lists of numbers and try to recognise a pattern if possible:

1, 3, 5, 7, 9 ..... (1)

2, 4, 6, 8, 10 ... (2)

1, 4, 7, 10, 13 .... (3)

5, 3, 1, –1, –3... (4)

1, 3, 9, 27, 81, ... (5)

2, 3, 5, 7, 11, 13... (6)

You can observe that numbers in the list (1) are odd natural numbers. The first number is 1, second number is 3, third number is 5, etc. All these numbers follow a pattern. The pattern is that all these numbers, except the first is obtained by adding 2 to its previous number.

In lists (2), (3) and (4), each number except the first is obtained by adding 2, 3, and – 2 respectively to its previous number.

In (5), each number, except the first is obtained by multiplying 3 to its previous number. In the list (6), you can see that it is the list of prime numbers and it is not possible to give any rule till date, which gives the next prime number.

The numbers in a list are generally denoted by

$a_1, a_2, a_3, \ldots, a_n, \ldots$

or $t_1, t_2, t_3, \ldots, t_n, \ldots$

which are respectively called first, second, third and nth term in the list of numbers. We sometimes call each of these lists as **sequence** or **pattern of numbers.**

7.2 ARITHMETIC PROGRESSION

You have seen different type of patterns. Some patterns follow definite mathematical rules to generate next term in the pattern. You will now study one particular type of pattern of numbers.Recall the following patterns.

1, 3, 5, 7, 9, ... (1)

2, 4, 6, 8, 10, ... (2)

1, 4, 7, 10, 13,... (3)

You have observed that in (1) and (2), each term except first is obtained by adding 2 to its previous number (term). In (3), each term except first is obtained by adding 3 to its previous term. The numbers appearing in a number pattern are called its terms. As already stated these terms are usually denoted by

$a_1, a_2, a_3, ..., a_n, ...$

or $t_1, t_2, t_3, ..., t_n, ...$ etc

The suffix denotes the position of the term in the pattern. Thus, a_n or t_n denotes 'n'th term of the pattern.

A particular type of pattern in which each term except the first is obtained by adding a fixed number (positive or negative) to the previous term is called an Arithmetic Progression (A.P.). The first term is usually denoted by '*a*' and the common difference is denoted by *d*. Thus, standard form of an Arithmetic Progression would be:

$a, a + d, a + 2d, a + 3d, ...$

Example 7.1: In the following list of numbers, find which are Arithmetic Progressions. In case of AP, find their respective first terms and common differences.

(i) 2, 7, 12, 17, 22, ...

(ii) 4, 0, –4, –8, –12 ...

(iii) 3, 7, 12, 18, 25 ...

(iv) 2, 6, 18, 54, 162 ...

Solution:

(i) It is an arithmetic progression (AP).

Since $7 - 2 = 5$, $12 - 7 = 5$, $17 - 12 = 5$ and $22 - 17 = 5$

Thus, each term except first is obtained by adding 5 to its previous term. Hence, first term a = 2 and common difference d = 5.

(ii) We observe that

$0 - 4 = -4$, $-4 - 0 = -4$, $-8 - (-4) = -4$, $-12 - (-8) = -4$

Thus, it is an AP with first term a = 4

and common difference d = – 4.

(iii) You can see that in the list

3, 7, 12, 18, 25, ...

7 – 3 = 4, 12 – 7 = 5, 18 – 12 = 6, 25 – 18 = 7

Thus, difference of two consecutive terms is not the same. Hence, it is not an AP.

(iv) In the list of numbers

2, 6, 18, 54, 162, ...

6 – 2 = 4, 18 – 6 = 12

Therefore, difference of two consecutive terms is not the same. Hence, it is not an AP.

CHECK YOUR PROGRESS 7.1

Which of the following are AP? If they are in AP, find their first terms and common differences:

1. –5, –1, 3, 7, 11, ...
2. 6, 7, 8, 9, 10, ...
3. 1, 4, 6, 7, 6, 4, ...
4. –6, – 3, 0, 3, 6, 9, ...

7.3 GENERAL (nth) TERM OF AN AP

Let us consider an AP whose first term is '*a*' and common difference in '*d*'. Let us denote the terms of AP as $t_1, t_2, t_3, ..., t_n$, where t_n denotes the nth term of the AP. Since first term is a, second term is obtained by adding d to a i.e., a + d, the third term will be obtained by adding '*d*' to a + d. So, third term will be (a + d) + d = a + 2d and so on.

With this

First term, t_1	$= a$	$= a + (1 - 1)\,d$
Second term, t_2	$= a + d$	$= a + (2 - 1)\,d$
Third term, t_3	$= a + 2d$	$= a + (3 - 1)\,d$
Fourth term, t_4	$= a + 3d$	$= a + (4 - 1)\,d$

Can you see any pattern? We observe that each term is a + (term number – 1) d. What will be 10th term, say:

$$t_{10} = a + (10 - 1)d = a + 9d$$

Can you now say "what will be the nth term or general term?"

Clearly $t_n = a + (n - 1)\,d$

Example 7.2: Find the 15th and nth terms of the AP

$$16, 11, 6, 1, -4, -9, \ldots$$

Solution: Here $a = 16$ and $d = 11 - 16 = -5$

Thus, $t_{15} = a + (15 - 1)d = a + 14d$

$= 16 + 14(-5) = 16 - 70$

$= -54$

Therefore, 15th term i.e., $t_{15} = -54$

Now $t_n = a + (n - 1)d$

$= 16 + (n - 1) \times (-5) = 16 - 5n + 5$

$= 21 - 5n$

Thus, nth term, i.e., $t_n = = 21 - 5n$

Example 7.3: The first term of an AP is -3 and 12th term is 41. Determine the common difference.

Solution: Let first term of AP be a and commong difference be d.

Therefore, $t_{12} = a + (12 - 1)d = 41$

or $-3 + 11d = 41$ [Since $a = -3$]

or $11d = 44$

or $d = 4$

Therefore, common difference is 4.

Example 7.4: The common difference of an AP is 5 and 10th term is 43. Find its first term.

Solution: We have:

$t_{10} = a + (10 - 1)\, d$

So, $43 = a + 9 \times 5$ [Since $d = 5$]

or $43 = a + 45$

Hence, $a = -2$

Therefore, first term is -2.

Example 7.5: The first term of an AP is -2 and 11th term is 18. Find its 15th term.

Solution: To find 15th term, you need to find d.

Now $t_{11} = a + (11 - 1)d$

So, $18 = -2 + 10d$

or $10d = 20$

or $d = 2$

Now $t_{15} = a + 14d$

$= -2 + 14 \times 2 = 26$

Therefore, $t_{15} = 26$.

Example 7.6: If p times the pth term of an AP is equal to q times the qth term, prove that its (p + q)th term is zero, provided $p \neq q$.

Solution: We have:

$t_p = a + (p-1)d$

$t_q = a + (q-1)d$

Since $pt_p = qt_q$, therefore,

$p[a + (p-1)d] = q[a + (q-1)d]$

or $pa + p(p-1)d - qa - q(q-1)d = 0$

or $(p-q)a + (p^2 - q^2)d - pd + qd = 0$

or $(p-q)a + (p^2 - q^2)d - (p-q)d = 0$

or $(p-q)a + (p-q)(p+q)d - (p-q)d = 0$

or $(p-q)[a + (p+q)d - d] = 0$

or $a + (p+q)d - d = 0$ $\quad$ [as $p - q \neq 0$]

or $a + (p + q - 1)d = 0$

Since, LHS is nothing but (p + q)th term, therefore,

$t_{p+q} = 0$

CHECK YOUR PROGRESS 7.2

1. The first term of an AP is 4 and common difference is – 3, find its 12th term.
2. The first term of an AP is 2 and 9th term is 26, find the common difference.
3. The 12th term of an AP is – 28 and 18th term is – 46. Find its first term and common difference.
4. Which term of the AP 5, 2, –1, ... is – 22?
5. If the *p*th, *q*th and *r*th terms of an AP are x, y and z respectively, prove that:

   $x(q - r) + y(r - p) + z(p - q) = 0$

7.4 SUM OF FIRST n TERMS OF AN AP

Carl Friedrich Gauss, the great German mathematician, was in elementary school, when his teacher asked the class to find the sum of first 100 natural numbers. While the rest of the class was struggling with the problem, Gauss found the answer within no time. How Gauss got the answer? Probably, he did as follows:

$$S = 1 + 2 + 3 + ... + 99 + 100 \qquad (1)$$

Writing these numbers in reverse order, we get

$$S = 100 + 99 + 98 + ... + 2 + 1 \qquad (2)$$

Adding (1) and (2), term by term, we get

$$2S = 101 + 101 + 101 + ... + 101 + 101 \text{ (100 times)}$$

$$= 100 \times 101$$

$$\text{or } S = \frac{100 \times 101}{2} = 5050$$

We shall use the same method to find the sum of first 'n' terms of an AP.

The first 'n' terms of an AP are

$$a, a + d, a + 2d, ..., a + (n - 2)d, a + (n - 1)d$$

Let us denote the sum of n terms by S_n. Therefore,

$$S_n = a + (a + d) + (a + 2d) + ... + [a + (n - 2)d] + [a + (n - 1)d] \qquad (3)$$

Writing these terms in reverse order, we get

$$S_n = [a + (n - 1)d] + [a + (n - 2)d] + ... + (a + d) + a \qquad (4)$$

We now add (3) and (4), term by term. We can see that the sum of any term in (3) and the corresponding term in (4) is $2a + (n - 1)d$. We get

$$2S_n = [2a + (n - 1)d] + [2a + (n - 1)d] + ... + [2a + (n - 1)d] + [2a + (n - 1)d], \text{ n times}$$

$$\text{or } 2S_n = n[2a + (n - 1)d]$$

$$\text{or } S_n = \frac{n}{2}[2a + (n - 1)d],$$

which gives general formula for finding the sum of first '*n*' terms of an AP.

This can be rewritten as

$$S_n = \frac{n}{2}[a + \{a + (n - 1)d\}]$$

$$= \frac{n}{2}(a + t_n), \qquad [\text{as } n^{th} \text{ term } t_n = a + (n - 1)d]$$

Sometimes, *n*th term is named as last term and is denoted by '*l*'. Thus:

$$S_n = \frac{n}{2}(a + l) \qquad (4)$$

Example 7.7: Find the sum of the first 12 terms of the following AP

(i) 11, 16, 21, 26 ....

(ii) – 151, – 148, – 145, – 142

Solution: (i) The given AP is

11, 16, 21, 26 ....

Here, $a = 11$, $d = 16 - 11 = 5$ and $n = 12$.

You know that sum of first n terms of an AP is given by

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

Therefore, $$S_{12} = \frac{12}{2}[2 \times 11 + (12-1)5]$$

$$= 6\,[22 + 55] = 6 \times 77 = 462$$

Hence, required sum is 462.

(ii) The given AP is

– 151, – 148, – 145, – 142

Here, $a = -151$, $d = -148 - (-151) = 3$ and $n = 12$.

We know that

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

Hence, sum of first 12 terms is

$$S_{12} = \frac{12}{2}[2 \times (-151) + (12-1)3]$$

$$= 6[-302 + 33] = 6 \times (-269)$$

$$= -1614$$

Therefore, required sum is – 1614.

Example 7.8: How may terms of the AP 2, 4, 6, 8, 10 .... are needed to get sum 210?

Solution: For the given AP, a = 2, d = 2 and $S_n = 210$.

We have: $$S_n = \frac{n}{2}[2a + (n-1)d]$$

or $$210 = \frac{n}{2}[2 \times 2 + (n-1)2]$$

or $$420 = n[2n + 2]$$

or $$420 = 2n^2 + 2n$$

or $$2n^2 + 2n - 420 = 0$$

or $$n^2 + n - 210 = 0$$

or $$n^2 + 15n - 14n - 210 = 0$$

or $$n(n + 15) - 14(n + 15) = 0$$

or $$(n + 15)(n - 14) = 0$$

or $$n = -15 \text{ or } n = 14$$

Since, *n* cannot be negative, so, n = 14

Therefore, first 14 terms are needed to get the sum 210.

Example 7.9: Find the following sum

$$2 + 5 + 8 + 11 + ... + 59$$

Solution: Here 2, 5, 8, 11, ... are in AP and a = 2, d = 3 and $t_n = 59$.

To find the sum, you need to find the value of n.

Now, $$t_n = a + (n-1)d$$

So, $$59 = 2 + (n-1)3$$

or $$59 = 3n - 1$$

or $$60 = 3n$$

Therefore, $$n = 20$$

Now, $$S_n = \frac{n}{2}[2a + (n-1)d]$$

or $$S_{20} = \frac{20}{2}[2 \times 2 + (20-1)3]$$

or $S_{20} = 10[4 + 57] = 610$

Therefore, required sum is 610.

Example 7.10: Find the sum of all natural numbers between 1 and 1000 which are divisible by 7.

Solution: Here, the first number which is divisible by 7 is 7 and last number, which is divisible by 7 is 994. Therefore, the terms to be added are

7, 14, 21, ..., 994

Here $a = 7$, $d = 7$, $t_n = 994$

Now $t_n = a + (n - 1)d$

or $994 = 7 + (n - 1)7$

or $994 = 7n$

This gives $n = 142$.

Now, $S_n = \frac{n}{2}[a + l]$

$$= \frac{142}{2}[7 + 994] = 71 \times 1001$$

$= 71071$

Therefore, required sum is 71071.

Example 7.11: The sum of first three terms of an AP is 36 and their product is 1620. Find the AP.

Solution: We can take three terms of the AP as a, a + d and a + 2d. However, the product will be rather difficult and solving the two equations simultaneoulsy will be time consuming. The elegant way is to assume the first three terms as a – d, a and a + d, so that the sum of three terms becomes 3a.

Let first three terms of the AP b a – d, a and a + d

Therefore, $a - d + a + a + d = 36$

or $3a = 36$,

which gives $a = 12$

Now, since product is 1620, we have:

$(a - d)\, a\, (a + d) = 1620$

or $(12 - d)\, 12\, (12 + d) = 1620$

or $12^2 - d^2 = 135$

or $\quad 144 - d^2 = 135$

or $\quad d^2 = 9$

Therefore, $\quad d = 3$ or -3

If d = 3, the numbers are 12 – 3, 12 and 12 + 3

i.e. 9, 12, 15 (Since a = 12)

If d = – 3, the numbers are15, 12 and 9

Therefore, the first three terms of the AP 9, 12, 15 and 15, 12, 9

satisfy the given conditions.

CHECK YOUR PROGRESS 7.3

1. Find the sum of first 15 terms of the following APs:

   (i) 11, 6, 1, – 4, –9 ...

   (ii) 7, 12, 17, 22, 27 ...

2. How many terms of the AP 25, 28, 31, 34, ... are needed to give the sum 1070?
3. Find the following sum:

   1 + 4 + 7 + 10 + ... +118

4. Find the sum of all natural numbers upto 100 which are divisible by 3.
5. The sum of any three consecutive terms of an AP is 21 and their product is 231. Find the three terms of the AP.
6. Of the l, a, n, d and S_n, determine the ones that are missing for each of the following arithmetic progression

   (i) $a = -2$, $d = 5$, $S_n = 568$.

   (ii) $l = 8$, $n = 8$, $S_8 = -20$

   (iii) a = – 3030, $l = -1530$, $n = 5$

   (iv) $d = \frac{2}{3}$, $l = 10$, $n = 20$

LET US SUM UP

- A progression in which each term, except the first, is obtained by adding a constant to the previous term is called an AP.
- The first term of an AP is denoted by a and common difference by d.

- The 'n'th term of an AP is given by $t_n = a + (n - 1)d$.
- The sum of first n terms of an AP is given by $S_n = \frac{n}{2}[2a + (n-1)d]$
- The sum of an AP whose first term is a and last term is l and number of terms is n is given by $S_n = \frac{n}{2}(a + l)$

TERMINAL EXERCISE

1. Which of the following patterns are arithmetic progressions?

   (i) 2, 5, 8, 12, 15, ...

   (ii) – 3, 0, 3, 6, 9 .....

   (iii) 1, 2, 4, 8, 16, ...

2. Write the nth term of each of the following arithmetic progressions:

   (i) 5, 9, 13, 17, ...

   (ii) – 7, – 11, – 15, – 19

3. The fourth term of an AP is equal to three times its first term and seventh term exceeds twice the third term by 1. Find the first term and common difference.

4. The 5th term of an AP is 23 and 12th term is 37. Find the first term and common difference.

5. The angles of a triangle are in AP. If the smallest angle is one-third the largest angle, find the angles of the triangle.

6. Which term of AP

   (i) 100, 95, 90, 85, ..., is – 25?

   (ii) $\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}.....$ is $\frac{25}{4}$?

7. The nth term of an AP is given by $t_n = a + bn$. Show that it is an AP. Find its first term and common difference.

8. If 7 times the 7th term of an AP is equal to 11 times the 11th term, show that the 18th term is zero.

9. Each term of an AP whose first term is a and common difference is d, is doubled. Is the resulting pattern an AP? If so, find its first term and common difference.

10. If k + 2, 4k – 6 and 3k – 2 are three consecutive terms of an AP, find k.

11. How many terms of the AP:

    (i) 1, 4, 7, 10, ... are needed to get the sum 715?

    (ii) –10, –7, –4, –1, ... are needed to get the sum 104?

12. Find the sum of first 100 odd natural numbers.

13. In an AP, a = 2 and sum of the first five terms is one-fourth the sum of the next five terms. Show that its 20th term is –12.

    [**Hint:** If AP is a, a + d, a + 2d, ... , then $S_5 = \frac{5}{2}[a + (a + 4d)]$

    In the next five terms, the first term is a + 5d and last term is a + 9d.

14. If sum of first n terms of an AP is $2n + 3n^2$, find rth term of the A.P. [Hint $t_r = S_r - S_{r-1}$]

15. Find the sum of all 3-digit numbers which leave the remainder 1, when divided by 4.

    [**Hint:** First term = 101, last term = 997]

ANSWERS TO CHECK YOUR PROGRESS

7.1

1. a = – 5, d = 4
2. a = 6, d = 1
3. Not an AP
4. a = –6, d = 3

7.2

1. – 29   2. 3   3. 5, – 3   4. 10^{th} term

7.3

1. (i) – 360   (ii) 630
2. 20
3. 2380
4. 1689
5. 3, 7, 11 or 11, 7, 3
6. (i) $n = 16, l = 73$ (ii) $a = -3, d = 3$

   (iii) $d = 375, S_n = -11400$ (iv) $a = -\frac{3}{8}, S_n = \frac{220}{3}$

ANSWERS TO TERMINAL EXERCISE

1. (ii)
2. (i) $t_n = 4n + 1$ (ii) $t_n = -4n - 3$
3. 3, 2
4. 15, 2
5. 30^o, 60^o, 90^o
6. (i) 26^{th} term (ii) 25^{th} term
7. a + b, b
9. Yes, first term = 2a, common difference = 2d
10. 3
11. (i) 22 terms (ii) 13 terms
12. 10,000
14. 6r – 1
15. 123525

Secondary Course Mathematics

Practice Work-Algebra

Maximum Marks: 25 **Time : 45 Minutes**

Instructions:

1. Answer all the questions on a separate sheet of paper.
2. Give the following informations on your answer sheet

   Name

   Enrolment number

   Subject

   Topic of practice work

   Address
3. Get your practice work checked by the subject teacher at your study centre so that you get positive feedback about your performance.

Do not send practice work to National Institute of Open Schooling

1. The value of a if $(x-a)$ is a factor of $x^6 - ax^5 + x^4 - ax^3 + 3x - a + 2$, is 1

   (A) $a = 1$

   (B) $a = -1$

   (C) $a = 2$

   (D) $a = -2$

2. The reciprocal of $\dfrac{1}{(-3/5)^{-2}}$ is 1

   (A) $\left(-\dfrac{3}{5}\right)^2$

   (B) $\left(\dfrac{-5}{3}\right)^2$

(C) $\left(-5/3\right)^{-2}$

(D) $\left(\dfrac{3}{5}\right)^{-2}$

3. In an A.P., the sum of three numbers is 15 and their product is 45. Then the three numbers are 1

(A) 1, 3, 15

(B) 2, 4, 9

(C) 1, 5, 9

(D) 0, 5, 9

4. If $y = \dfrac{x-1}{x+1}$, then $2y - \dfrac{1}{2y}$ is equal to 1

(A) $\dfrac{3x^2-10x-3}{2\left(x^2-1\right)}$

(B) $\dfrac{3x^2-10x+1}{x^2-1}$

(C) $\dfrac{3x^2+10x+3}{2\left(x^2-1\right)}$

(D) $\dfrac{3x^2-10x+3}{2\left(x^2-1\right)}$

5. The lowest form of the expression $\dfrac{4x^2-25}{2x^2+11x=15}$ is 1

(A) $\dfrac{2x-5}{x+3}$

(B) $\dfrac{2x+5}{x+3}$

(C) $\dfrac{2x-5}{x-3}$

(D) $\dfrac{2x-5}{x-3}$

6. Find x, so that $\left(\dfrac{7}{8}\right)^{-3} \times \left(\dfrac{8}{7}\right)^{-11} = \left(\dfrac{7}{8}\right)^{x}$: 2

7. Find three irrational numbers between $\sqrt{3}$ and $\sqrt{8}$. 2

8. The HCF of two polynomials is (x–2) and their LCM is $x^4 + 2x^3 - 8x - 16$. If one of the polynomials is $x^3 - 8$, find the other polynomial. 2

9. The sum of a number and its reciprocal is $\dfrac{50}{7}$, find the number. 2

10. The length of a rectangle is 5 cm less than twice its breadth. If the perimeter is 110 cm, find the area of the rectangle. 2

11. Show that the sum of an AP whose first term is a, the second term is b and the last term is c, is equal to $\dfrac{(a+c)(b+c-2a)}{2(b-a)}$. 4

12. Had Ajay scored 10 more marks in his test out of 30 marks, 9 times these marks would have been the square of his actual marks. How many marks did he get in the test? 6

MODULE 2

Commercial Mathematics

It is a common saying by elders keep your expenditure, less than your income. The latent meaning of this is to save something for difficult times. You must have seen birds and animals saving eatables for rainy season, in their nests or caves. Taking the lead from this, the students have been told about the importance and need of savings in this module

Many Indian mathematicians have worked on the topic of commercial Mathematics. Yodoksu (370 B.C.) worked on fractions and ratio and proportion. In the reigns of Ashoka and Chandragupta, there is a description of levying taxes. There is a description of many mathematicians working on practice and proportion (like Aryabhatt, Mahavira, Brahmgupta, Sridharacharya). In 900 A.D., Bakhshali Manuscript was discovered which had a number of problems on Commercial mathematics.

To keep your savings safe is another tough task. Banks and other financial institutions keep the money of their customers and on the expiry of the period pay extra money, called interest, in addition to the money deposited. This encourages citizens to save and keep the money safe. This is why calculation of interest on deposits in banks is included for teaching.

The Government provides a number of facilities to the citizens. For that they levy certain taxes on citizens. One of these taxes is sales tax to which the learners are introduced in this "module. Financial transactions about buying and selling are generally done for profit. Due to greater supply of goods or sub-standard goods they are to be sold on loss. The learners are, therefore, introduced to percentage and profit and loss. Sometimes we have to buy articles on instalments because of non-availability of adequate funds. Due to this the students are taught to calculate interest when they buy articles on instalment plan. Sometimes when we are not able to return loaned money on time, the financer starts charging interest on interest also, which is called compound interest. Due to this the study of compound interest has been included in this module. The formulae of compound interest is also used in finding increase or decrease in prices of things. This is also taught under "Appreciation and Depreciation" of value.

8

211en08

PERCENTAGE AND ITS APPLICATIONS

You must have seen advertisements in newspapers, television and hoardings etc of the following type:

"Sale, up to 60% off".

"Voters turnout in the poll was over 70%".

"Ramesh got 93% aggregate in class XII examination".

"Banks have lowered the rate of interest on fixed deposits from 8.5% to 7%".

In all the above statements, the important word is 'percent'. The word 'percent' has been derived from the Latin word 'percentum' meaning per hundred or out of hundred.

In this lesson, we shall study percent as a fraction or a decimal and shall also study its applications in solving problems of profit and loss, discount, simple interest, compound interest, rate of growth and depreciation etc.

OBJECTIVES

After studying this lesson, you will be able to

- *illustrate the concept of percentage;*
- *calculate specified percent of a given number or a quantity;*
- *solve problems based on percentage;*
- *solve problems based on profit and loss;*
- *calculate the discount and the selling price of an article, given marked price of the article and the rate of discount;*
- *solve inverse problems pertaining to discount;*
- *calculate simple interest and the amount, when a given sum of money is invested for a specified time period on a given rate of interest;*

- *illustrate the concept of compound interest vis-a-vis simple interest;*
- *calculate compound interest, the amount and the difference between compound and simple interest on a given sum of money at a given rate and for a given time period; and*
- *solve real life problems pertaining to rate of growth and decay, using the formula of compound interest, given a uniform or variable rate.*

EXPECTED BACKGROUND KNOWLEDGE

- Four fundamental operations on whole numbers, fractions and decimals.
- Comparison of two fractions.

8.1 PERCENT

Recall that a fraction $\frac{3}{4}$ means 3 out of 4 equal parts. $\frac{7}{13}$ means 7 out of 13 equal parts and $\frac{23}{100}$ means 23 out of 100 equal parts.

A fraction whose denominator is 100 is read as percent, for example $\frac{23}{100}$ is read as twenty three percent.

The symbol '%' is used for the term percent.

A ratio whose second term is 100 is also called a percent,

So, 33 : 100 is equivalent to 33%.

Recall that while comparing two fractions, $\frac{3}{5}$ and $\frac{1}{2}$, we first convert them to equivalent fractions with common denominator (L.C.M. of the denominators).

thus $\frac{3}{5}=\frac{3}{5}\times\frac{2}{2}=\frac{6}{10}$, and

$$\frac{1}{2}=\frac{1}{2}\times\frac{5}{5}=\frac{5}{10}$$

Now, because $\frac{6}{10}>\frac{5}{10}$ $\therefore \frac{3}{5}>\frac{1}{2}$

We could have changed these fractions with common denominator 100 as

$$\frac{3}{5}=\frac{3}{5}\times\frac{20}{20}=\frac{60}{100} \quad \text{or} \quad 60\%$$

$$\frac{1}{2}=\frac{1}{2}\times\frac{50}{50}=\frac{50}{100} \quad \text{or} \quad 50\%$$

and so, $\frac{3}{5}>\frac{1}{2}$ as 60% is greater than 50%.

8.2 CONVERSION OF A FRACTION INTO PERCENT AND VICE VERSA

In the above section, we have learnt that, to convert a fraction into percent, we change the fraction into an equivalent fraction with denominator 100 and then attach the symbol % with the changed numerator of the fraction. For example,

$$\frac{3}{4}=\frac{3}{4}\times\frac{25}{25}=\frac{75}{100}=75\times\frac{1}{100}=75\% \quad \text{and}$$

$$\frac{4}{25}=\frac{4}{25}\times\frac{4}{4}=\frac{16}{100}=16\times\frac{1}{100}=16\%$$

Note: To write a fraction as percent, we may multiply the fraction by 100, simplify it and attach % symbol. For example,

$$\frac{4}{25}=\frac{4}{25}\times 100\%=16\%$$

Conversely,

To write a percent as a fraction, we drop the % sign, multiply the number by $\frac{1}{100}$ (or divide the number by 100) and simplify it. For example,

$$47\%=47\times\frac{1}{100}=\frac{47}{100}, \qquad 17\%=17\times\frac{1}{100}=\frac{17}{100}, \qquad 3\%=\frac{3}{100}$$

$$45\%=45\times\frac{1}{100}=\frac{45}{100}=\frac{9}{20}, \quad 210\%=\frac{210}{100}=\frac{21}{10}, \quad x\%=\frac{x}{100}.$$

8.3 CONVERSION OF DECIMAL INTO A PERCENT AND VICE VERSA

Let us consider the following examples:

$$0.35 = \frac{35}{100} = 35 \times \frac{1}{100} = 35\%$$

$$4.7 = \frac{47}{10} = \frac{470}{100} = 470 \times \frac{1}{100} = 470\%$$

$$0.459 = \frac{459}{1000} = \frac{459}{10} \times \frac{1}{100} = 45.9\%$$

$$0.0063 = \frac{63}{10000} = \frac{63}{100} \times \frac{1}{100} = 0.63\%$$

Thus, **to write a decimal as a percent, we move the decimal point two places to the right and put the % sign**

Conversely,

To write a percent as a decimal, we drop the % sign and insert or move the decimal point two places to the left. For example,

$43\% = 0.43$	$75\% = 0.75$	$12\% = 0.12$
$9\% = 0.09$	$115\% = 1.15$	$327\% = 3.27$
$0.75\% = 0.0075$	$4.5\% = 0.045$	$0.2\% = 0.002$

Let us take a few more examples:

Example 8.1: Shweta obtained 18 marks in a test of 25 marks. What was her percentage of marks?

Solution: Total marks = 25

Marks obtained = 18

$\therefore$ Fraction of marks obtained $= \frac{18}{25}$

$\therefore$ Marks obtained in percent $= \frac{18}{25} \times \frac{4}{4} = \frac{72}{100} = 72\%$

Alternatively:

$$\text{Marks obtained in percent} = \frac{18}{25} \times 100\% = 72\%$$

Example 8.2: One-fourth of the total number of shoes in a shop were on discount sale. What percent of the shoes were there on normal price?

Solution: Fraction of the total number of shoes on sale $= \frac{1}{4}$

$\therefore$ Fraction of the total number of shoes on normal price $= 1 - \frac{1}{4} = \frac{3}{4}$

$$= \frac{3}{4} \times \frac{25}{25} = \frac{75}{100} = 75\% \text{ or } \frac{3}{4} \times 100\% = 75\%$$

Example 8.3: Out of 40 students in a class, 32 opted to go for a picnic. What percent of students opted for picnic?

Solution: Total number of students in a class = 40

Number of students, who opted for picnic = 32

$\therefore$ Number of students, in percent, who opted for picnic

$$= \frac{32}{40} \times 100\% = 80\%$$

Example 8.4: In the word ARITHMETIC, what percent of the letters are I's?

Solution: Total number of letters = 10

Number of I's = 2

$\therefore$ Percent of I's $= \frac{2}{10} \times 100\% = 20\%$

Example 8.5: A mixture of 80 litres, of acid and water, contains 20 litres of acid. What percent of water is in the mixture?

Solution: Total volume of the mixture = 80 litres

Volume of acid = 20 litres

$\therefore$ Volume of water = 60 litres

$\therefore$ Percentage of water in the mixture $= \frac{60}{80} \times 100\% = 75\%$

CHECK YOUR PROGRESS 8.1

1. Convert each of the following fractions into a percent:

(a) $\frac{12}{25}$ (b) $\frac{9}{20}$ (c) $\frac{5}{12}$ (d) $\frac{6}{15}$ (e) $\frac{125}{625}$

(f) $\frac{3}{10}$ (g) $\frac{108}{300}$ (h) $\frac{189}{150}$ (i) $\frac{72}{25}$ (j) $\frac{1231}{1250}$

2. Write each of the following percents as a fraction:

(a) 53% (b) 85% (c) $16\frac{7}{8}\%$ (d) 3.425% (e) 6.25%

(f) 70% (g) $15\frac{3}{4}\%$ (h) 0.0025% (i) 47.35% (j) 0.525%

3. Write each of the following decimals as a percent:

(a) 0.97 (b) 0.735 (c) 0.03 (d) 2.07 (e) 0.8

(f) 1.75 (g) 0.0250 (h) 3.2575 (i) 0.152 (j) 3.0015

4. Write each of the following percents as a decimal:

(a) 72% (b) 41% (c) 4% (d) 125% (e) 9%

(f) 410% (g) 350% (h) 102.5% (i) 0.025% (j) 10.25%

5. Gurpreet got half the answers correct, in an examination. What percent of her answers were correct?

6. Prakhar obtained 18 marks in a test of total 20 marks. What was his percentage of marks?

7. Harish saves ₹ 900 out of a total monthly salary of ₹ 14400. Find his percentage of saving.

8. A candidate got 47500 votes in an election and was defeated by his opponent by a margin of 5000 votes. If there were only two candidates and no votes were declared invalid, find the percentage of votes obtained by the winning candidate.

9. In the word PERCENTAGE, what percent of the letters are E's?

10. In a class of 40 students, 10 secured first division, 15 secured second division and 13 just qualified. What percent of students failed.

8.4 CALCULATION OF PERCENT OF A QUANTITY OR A NUMBER

To determine a specified percent of a number or quantity, we first change the percent to a fraction or a decimal and then multiply it with the number or the quantity. For example:

$$25\% \text{ of } 90 = \frac{25}{100} \times 90 = 22.50$$

or $25\% \text{ of } 90 = 0.25 \times 90 = 22.50$

$60\% \text{ of Rs. } 120 = 0.60 \times \text{Rs. } 120 = \text{Rs. } 72.00$

$120\% \text{ of } 80 \text{ kg} = 1.20 \times 80 \text{ kg} = 96 \text{ kg}$

Let us take some examples from daily life:

Example 8.6: In an examination, Neetu scored 62% marks. If the total marks in the examination are 600, then what are the marks obtained by Neetu?

Solution: Here we have to find 62% of 600

$\therefore$ 62% of 600 marks = 0.62 × 600 marks = 372 marks

$\therefore$ Marks obtained by Neetu = 372

Example 8.7: Naresh earns ₹ 30800 per month. He keeps 50% for household expenses, 15% for his personal expenses, 20% for expenditure on his children and the rest he saves. What amount does he save per month?

Solution: Expenditure on Household = 50%

Expenditure on self = 15%

Expenditure on children = 20%

Total expenditure = (50 + 15 + 20)% = 85%

$\therefore$ Savings (100 – 85)% = 15%

$\therefore$ 15% of ₹ 30800 = ₹ (0.15 × 30800)

= ₹ 4620

Example 8.8: What percent of 360 is 144?

Solution: Let x% of 360 = 144

$$\therefore \frac{x}{100} \times 360 = 144$$

Or $$x = \frac{144}{360} \times 100 = 40\%$$

Alternatively, 144 out of 360 is equal to the fraction $\frac{144}{360}$

$$\therefore \text{Percent} = \frac{144}{360} \times 100\% = 40\%$$

Example 8.9: If 120 is reduced to 96, what is the reduction percent?

Solution: Here, reduction = 120 – 96 = 24

$$\therefore \text{Reduction percent} = \frac{24}{120} \times 100\% = 20\%$$

Example 8.10: The cost of an article has increased from ₹ 450 to ₹ 495. By what percent did the cost increased?

Solution: The increase in Cost Price = ₹ (495 – 450)

= ₹ 45

$$\text{Increase percent} = \frac{45}{450} \times 100 = 10\%$$

Example 8.11: 60% of the students in a school are girls. If the total number of girls in the school is 690, find the total number of students in the school. Also, find the number of boys in the school.

Solution: Let the total number of students in the school be x

Then, 60% of $x = 690$

$$\therefore \frac{60}{100} \times x = 690 \text{ or } x = \frac{690 \times 100}{60} = 1150$$

$\therefore$ Total number of students in the school = 1150

$\therefore$ Hence number of boys = 1150 – 690 = 460

Example 8.12: A's income is 25% more than that of B. B's income is 8% more than that of C. If A's income is ₹ 20250, then find the income of C.

Solution: Let income of C be ₹ x

Income of B $= x + 8\%$ of x

$$= x + \frac{8x}{100} = \frac{108}{100} \times x$$

$$\text{Income of A} = \frac{108x}{100} + 25\% \text{ of } \frac{108x}{100}$$

$$= \frac{108x}{100} \times \frac{125}{100}$$

$$\therefore \frac{108}{100} \times x \times \frac{125}{100} = 20250$$

$$\text{or } x = 20250 \times \frac{100}{108} \times \frac{100}{125} = 15000$$

$\therefore$ Income of C is ₹ 15000.

Example 8.13: A reduction of 10% in the price of tea enables a dealer to purchase 25 kg more tea for ₹ 22500. What is the reduced price per kg of tea? Also, find the original price per kg.

Solution: 10% of ₹ 22500 = $\frac{10}{100} \times 22500$ = ₹ 2250

$\therefore$ Reduced price of 25 kg tea = ₹ 2250

$\therefore$ Reduced price per kg = ₹ $\frac{2250}{25}$ = ₹ 90 per kg.

Since, the reduction was 10% so the original price = ₹ 100 per kg.

Example 8.14: A student got 45% marks in the first paper and 70% in the second paper. How much percent should he get in the third paper so as to get 60% as overall score?

Solution: Let each paper be of 100 marks.

$\therefore$ Marks obtained in first paper = 45% of 100 = 45

Marks obtained in second paper = 70% of 100 = 70

Total marks (in three papers) he wants to obtain = 60% of 300

$$= \frac{60}{100} \times 300 = 180$$

$\therefore$ Marks to be obtained in third paper = 180 – (45 + 70)

= 180 – 115 = 65

Example 8.15: Find the sum which when increased by 15% becomes ₹ 19320.

Solution: Let the sum be ₹ x

$\therefore$ x + 15% of x = 19320

$$x + \frac{15x}{100} = 19320 \quad \text{or} \quad \frac{115x}{100} = 19320$$

$$\therefore x = \frac{19320 \times 100}{115} = 16800$$

Hence, the required sum = ₹ 16800.

CHECK YOUR PROGRESS 8.2

1. Find: (i) 16% of 1250 (ii) 47% of 1200
2. A family spends 35% of its monthly budget of ₹ 7500 on food. How much does the family spend on food?

3. In a garden, there are 500 plants of which 35% are trees, 20% are shrubs, 25% are herbs and the rest are creepers. Find out the number of each type of plants.
4. 60 is reduced to 45. What percent is the reduction?
5. If 80 is increased to 125, what is the increase percent?
6. Raman has to score a minimum 40% marks for passing the examination. He gets 178 marks and fails by 22 marks. Find the maximum marks.
7. It takes me 45 minutes to go to school and I spend 80% of the time travelling by bus. How long does the bus journey last?
8. In an election, between 2 candidates 25% voters did not cast their votes. A candidate scored 40% of the votes polled and was defeated by 900 votes. Find the total number of voters.
9. A rise of 25% in the price of sugar compels a person to buy 1.5 kg of sugar less for ₹ 240. Find the increased price as well as the original price per kg of sugar.
10. A number is first increased by 20% and then decreased by 20%. What is the net increase or decrease percent?
11. 'A' scored 12 marks, while B scored 10 marks, in the first terminal examination. If in the second terminal examination (with same total number of marks) 'A' scored 14 marks and 'B' scored 12 marks, which student showed more improvement?
12. 30,000 students appeared in a contest. Of them 40% were girls and the remaining boys. If 10% boys and 12% girls won the contest with prizes, find the percentage of students who won prizes.
13. Sunil earns 10% more than Shailesh and Shailesh earns 20% more than Swami. If Swami earns ₹ 3200 less than Sunil, find the earnings of each.

8.5 APPLICATION OF PERCENTAGE

In our day to day life, we come across a number of situations wherein we use the concept of percent. In the following section, we discuss the application of percentage in different fields, like problems in profit and loss, discount, simple interest, compound interest, rate of growth and depreciation.

8.5.1 Profit and Loss

Let us recall the terms and formulae related to profit and loss.

Cost Price (C.P.): The Price at which an article is purchased, is called its cost price.

Selling Price (S.P.): The Price at which an article is sold, is called its selling price.

Profit (Gain): When S.P. > C.P., then there is profit, and

$$\text{Profit} = \text{S.P.} - \text{C.P.}$$

Loss: When C.P. > S.P., then there is loss, and

$$\text{Loss} = \text{C.P.} - \text{S.P.}$$

Formulae $\quad \text{Profit \%} = \left(\dfrac{\text{Profit}}{\text{C.P.}} \times 100\right)\%, \qquad \text{Loss\%} = \left(\dfrac{\text{Loss}}{\text{C.P.}} \times 100\right)\%$

$$\text{S.P.} = \frac{(\text{C.P.}) \times (100 + \text{Profit\%})}{100} = \frac{(\text{C.P.})(100 - \text{Loss\%})}{100}$$

$$\text{C.P.} = \frac{\text{S.P.} \times 100}{(100 + \text{Profit\%})} = \frac{(\text{S.P.}) \times 100}{(100 - \text{Loss\%})}$$

Note: Gain % or loss % is always calculated on C.P.

Let us take some examples to illustrate the applications of these formulae in solving problems related to profit and loss:

Example 8.16: A shopkeeper buys an article for Rs. 360 and sells it for Rs. 270. Find his gain or loss percent.

Solution: Here C.P. = Rs. 360, and S.P. = Rs. 270

Since C.P. > S.P., $\therefore$ there is a loss.

Loss = C.P. – S.P. = Rs (360 – 270) = Rs. 90

$$\text{Loss \%} = \left(\frac{\text{Loss}}{\text{C.P.}} \times 100\right)\%$$

$$= \frac{90}{360} \times 100 = 25\%$$

Example 8.17: Sudha purchased a house for ₹ 4,52,000 and spent ₹ 28,000 on its repairs. She had to sell it for ₹ 4,92,000. Find her gain or loss percent.

Solution: Here C.P. = Cost price + Overhead charges

= ₹ (452000 + 28000) = ₹ 4,80,000

S.P. = ₹ 4,92,000

Since, S.P. > C.P., $\therefore$ Gain = ₹ (492000 – 480000) = ₹ 12000

$$\text{Gain \%} = \frac{12000 \times 100}{480000} = \frac{5}{2}\% = 2.5\%$$

Example 8.18: By selling a book for ₹ 258, a publisher gains 20%. For how much should he sell it to gain 30%?

Solution: S.P. = Rs. 258

Profit = 20%

$$\text{C.P.} = \frac{\text{S.P.}\times 100}{100+\text{Profit\%}} = ₹\ \frac{258\times 100}{120} = ₹\ 215$$

Now, if Profit = 30% and C.P. = Rs. 215, then,

$$\text{S.P.} = \frac{\text{C.P.}\times(100+\text{Profit\%})}{100} = ₹\ \frac{215\times 130}{100} = ₹\ 279.50$$

Example 8.19: A man bought oranges at 25 for ₹ 100 and sold them at 20 for ₹ 100. Find his gain or loss percent.

Solution: C.P. of 25 oranges = ₹ 100

$$\therefore \text{ C.P. of 1 orange} = ₹\ \frac{100}{25} = ₹\ 4$$

$$\text{and S.P. of 1 orange} = ₹\ \frac{100}{20} = ₹\ 5$$

$\therefore$ Profit on 1 orange = ₹ (5 – 4) = ₹ 1

$$\text{Profit \%} = \frac{1}{4}\times 100 = 25\%$$

Example 8.20: A man sold two horses for ₹ 29700 each. On one he lost 10% while he gained 10% on the other. Find his total gain or loss percent in the transaction.

Solution: S.P. of first horse = ₹ 29700

Loss = 10%

$$\therefore \text{ C.P.} = ₹\ \frac{29700\times 100}{90} = ₹\ 33{,}000$$

S.P. of 2nd horse = ₹ 29700,

Profit = 10%

$$\text{C.P.} = ₹\ \frac{29700\times 100}{110} = ₹\ 27{,}000$$

Total CP = ₹ (33000 + 27000) = ₹ 60,000

Total SP = ₹ (2 × 29700) = ₹ 59400

Net Loss = ₹ (60000 – 59400) = ₹ 600

$$\text{Loss \%} = \frac{600}{60000} \times 100 = 1\%$$

Example 8.21: The cost price of 15 articles is equal to the selling price of 12 articles. Find the gain percent.

Solution: Let the C.P. of 15 articles be ₹ 15

then S.P. of 12 articles = ₹ 15

$$\text{S.P. of 15 articles} = ₹\ \frac{15}{12} \times 15 = ₹\ \frac{75}{4}$$

$$\text{Gain} = ₹\ \left(\frac{75}{4} - 15\right) = ₹\ \frac{15}{4}$$

$$\text{Gain \%} = \frac{15/4}{15} \times 100 = 25\%$$

Example 8.22: A watch was sold at a profit of 12%. Had it been sold for ₹ 33 more, the profit would have been 14%. Find the cost price of the watch.

Solution: Let the cost price of the watch be ₹ x

$$\therefore \text{S.P.} = \frac{x \times 112}{100} = \frac{112x}{100}$$

$$\text{If the watch is sold for Rs. 33 more then S.P.} = \left(\frac{112x}{100} + 33\right)$$

New profit = 14%

$$\therefore \text{C.P.} = x = \frac{\left(\frac{112x}{100} + 33\right) \times 100}{114}$$

or $114x = 112\,x + 3300$ or $2x = 3300$

$x = 1650$ $\therefore$ C.P. = ₹ 1650

CHECK YOUR PROGRESS 8.3

1. A shopkeeper bought an almirah from a wholesale dealer for ₹ 4500 and sold it for ₹ 6000. Find his profit or loss percent.

2. A retailer buys a cooler for ₹ 3800 but had to spend ₹ 200 on its transport and repair. If he sells the cooler for ₹ 4400, determine, his profit percent.
3. A vendor buys lemons at the rate of 5 for ₹ 7 and sells them at ₹ 1.75 per lemon. Find his gain percent.
4. A man purchased a certain number of oranges at the rate of 2 for ₹ 5 and sold them at the rate of 3 for ₹ 8. In the process, he gained ₹ 20. Find the number of oranges he bought.
5. By selling a bi-cycle for ₹ 2024, the shopkeeper loses 12%. If he wishes to make a gain of 12% what should be the selling price of the bi-cycle?
6. By selling 45 oranges for ₹ 160, a woman loses 20%. How many oranges should she sell for ₹ 112 to gain 20% on the whole?
7. A dealer sold two machines at ₹ 2400 each. On selling one machine, he gained 20% and on selling the other, he lost 20%. Find the dealer's net gain or loss percent.
8. Harish bought a table for ₹ 960 and sold it to Raman at a profit of 5%. Raman sold it to Mukul at a profit of 10%. Find the money paid by Mukul for the table.
9. A man buys bananas at 6 for ₹ 5 and an equal number at ₹ 15 per dozen. He mixes the two lots and sells them at ₹ 14 per dozen. Find his gain or loss percent, in the transaction.
10. If the selling price of 20 articles is equal to the cost price of 23 articles, find the loss or gain percent.

8.5.2 Discount

You must have seen advertisements of the following types, especially during the festival season.

SALE	}	DIWALI BONANZA
discount upto 50%		20% discount on all items.

A discount is a reduction in the marked (or list) price of an article. "20% discount"means a reduction of 20% in the marked price of an article. For example, if the marked price of an article is ₹ 100, it is sold for ₹ 80, i.e. ₹ 20 less than the marked price. Let us define the terms, we shall use:

Marked Price (or List price): The marked price (M.P.) of an article is the price at which the article is listed for sale. Since this price is written (marked) on the article, so it is called the marked price.

Discount: The discount is the reduction from the marked price of the article.

Net selling price (S.P.): In case of discount selling, the price of the article obtained by subtracting discount from the marked price is called the Net Selling price or Selling price (S.P.). Let us take the following examples, to illustrate:

Example 8.23: A coat is marked at ₹ 2400. Find its selling price if a discount of 12% is offered.

Solution: Here, Marked Price (M.P.) of the coat = ₹ 2400

Discount = 12%

Net selling price (S.P.) = M.P. – Discount

$$= ₹\ 2400 - 12\%\ \text{of}\ ₹\ 2400$$

$$= ₹\ 2400 - ₹\left(\frac{12}{100}\times 2400\right)$$

$$= ₹\ (2400 - 288)$$

$$= ₹\ 2112$$

Thus, the net selling price of coat is ₹ 2112.

Example 8.24: A machine listed at ₹ 8400 is available for ₹ 6300. Find the rate of discount offered.

Solution: Here, Marked Price (M.P.) = ₹ 8400

Net selling price (S.P.) = ₹ 6300

Discount offered = ₹ (8400 – 6300)

= ₹ 2100

$$\text{Discount }\% = \frac{2100}{8400}\times 100\% = 25\%$$

Note: Discount is always calculated on Marked Price.

Example 8.25: A wholesaler's list price of a fan is ₹ 1250 and is available to a retailer at a discount of 20%. For how much should the retailer sell it, to earn a profit of 15%.

Solution: M.P. = ₹ 1250

Discount = 20% of ₹ 1250

$$= ₹\ \frac{20}{100}\times 1250 = ₹\ 250$$

∴ Cost Price of the retailer = ₹ (1250 – 250)

= ₹ 1000

Profit = 15%

$$\therefore\ \text{S.P.} = \frac{\text{C.P.}(100+\text{Profit}\%)}{100} = ₹\ \frac{1000\times 115}{100}$$

$$= ₹\ 1150$$

Example 8.26: A shopkeeper marks his goods 25% more than their cost price and allows a discount of 10%. Find his gain or loss percent.

Solution: Let the C.P. of an article = ₹ 100

$\therefore$ Marked Price (M.P.) = ₹ 100 + 25% of ₹ 100

= ₹ 125

Discount offered = 10%

$\therefore$ Net selling Price = ₹ 125 – 10% of ₹ 125

$$= ₹\ 125 - ₹\left(\frac{10}{100}\times 125\right)$$

= ₹ (125 – 12.50) = ₹ 112.50

$\therefore$ Gain = ₹ (112.50 – 100) = ₹ 12.50

$$\text{Gain } \% = \frac{12.50}{100}\times 100 = 12.5\%$$

Example 8.27: An article listed at ₹ 5400 is offered at a discount of 15%. Due to festival season, the shopkeeper allows a further discount of 5%. Find the selling price of the article.

Solution: M.P. = ₹ 5400, Discount = 15%

$\therefore$ SP = ₹ 5400 – 15% of ₹ 5400

$$= ₹\ 5400 - ₹\ \frac{15}{100}\times 5400$$

= ₹ (5400 – 810) = ₹ 4590

Festival discount = 5%

$\therefore$ Net selling Price = ₹ 4590 – 5% of ₹ 4590

$$= ₹\ 4590 - ₹\ \frac{5}{100}\times 4590$$

= ₹ (4590 – 229.50)

= ₹ 4360.50

$\therefore$ Net selling price of article = ₹ 4360.50.

Example 8.28: A retailer buys books from a wholesaler at the rate of ₹ 300 per book and marked them at ₹ 400 each. He allows some discount and gets a profit of 30% on the cost price. What percent discount does he allow to his customers?

Solution: C.P. of one book = ₹ 300

M.P. = ₹ 400

Profit = 30%

$$\therefore \text{ S.P.} = \frac{\text{C.P.}(100 + \text{Profit}\%)}{100} = ₹\ \frac{300 \times 130}{100} = ₹\ 390$$

$$\therefore \text{ Discount offered} = ₹\ (400 - 390) = ₹\ 10$$

$$\text{Discount } \% = \frac{10}{400} \times 100 = 2.5\%$$

CHECK YOUR PROGRESS 8.4

1. A shirt with marked price ₹ 375/- is sold at a discount of 15%. Find its net selling price.
2. A pair of socks marked at ₹ 60 is being offered for ₹ 48. Find the discount percent being offered.
3. A washing machine is sold at a discount of 10% on its marked price. A further discount of 5% is offered for cash payment. Find the selling price of the washing machine if its marked price is ₹ 18000.
4. A man pays ₹ 2100 for a machine listed at ₹ 2800. Find the rate of discount offered.
5. The list price of a table fan is ₹ 840 and it is available to a retailer at a discount of 25%. For how much should the retailer sell it to earn a profit of 15%.
6. A shopkeeper marks his goods 50% more than their cost price and allows a discount of 40%, find his gain or loss percent.
7. A dealer buys a table listed at ₹ 2500 and gets a discount of 28%. He spends ₹ 100 on transportation and sells it at a profit of 15%. Find the selling price of the table.
8. A retailer buys shirts from a manufacturer at the rate of ₹ 175 per shirt and marked them at ₹ 250 each. He allows some discount and earns a profit of 28% on the cost price. What percent discount does he allow to his customers?

8.5.3 Simple Interest

When a person has to borrow some money as a loan from his friends. relatives, bank etc. he promises to return it after a specified time period along with some extra money for using the money of the lender.

The money borrowed is called the **Principal**, usually denoted by P, and the extra money paid is called the **Interest**, usually denoted by I.

The total money paid back, that is, the sum of Principal and the Interest is called the **Amount**, and is usually denoted by A.

Thus, $A = P + I$

The interest is mostly expresed as a rate percent per year (per annum).

Interest depends on, how much money (P) has been borrowed and the duraton of time (T) for which it is used. Interest is calculated according to a mutually agreed rate percent, per annum (R). [i.e. $R = r\,\% = \frac{r}{100}$]

Thus, Interest = (Principal) × (Rate % per annum) × time

$$\therefore I = P \times R \times T$$

Interest calculated as above, is called simple interest. Let us take some examples, involving simple interest.

Example 8.29: Find the simple interest in each of the following cases

	P	R	T
(a)	₹ 8000	5%	2 yrs
(b)	₹ 20,000	15%	$1\frac{1}{2}$ yrs

Solution: (a) I = P. R. T.

$$= ₹\left[8000 \times \frac{5}{100} \times 2\right] = ₹\ 800$$

(b) $I = ₹\left[20000 \times \frac{15}{100} \times \frac{3}{2}\right] = ₹\ 4500$

Example 8.30: Find at what rate of simple interest per annum will ₹ 5000 amount to ₹ 6050 in 3 years.

Solution: Here A = ₹ 6050, P = ₹ 5000, T = 3 yrs

$\therefore$ I = ₹ (6050 – 5000) = ₹ 1050

$I = P \times R \times T$ or $r\% = \frac{I}{P \times T}$ $\qquad \therefore r = \frac{I \times 100}{P \times T}$

$$r = \frac{1050 \times 100}{5000 \times 3} = 7 \qquad \therefore R = 7\%$$

Notes

Example 8.31: A sum amounts to ₹ 4875 at $12\frac{1}{2}\%$ simple interest per annum after 4 years. Find the sum.

Solution: Here A = ₹ 4875, $R = 12\frac{1}{2}\% = \frac{25}{2}\%$, T = 4 yrs

$$I = P \times R \times T$$

$$I = ₹\left(P \times \frac{25}{200} \times 4\right) = ₹\ \frac{P}{2}$$

$$\therefore\ A = ₹\left(P + \frac{P}{2}\right) = ₹\ \frac{3P}{2}$$

Thus, $\frac{3P}{2} = ₹\ 4875$ or $3P = ₹\ 9750$ or $P = ₹\ 3250$

Example 8.32: In how many years will a sum of ₹ 2000 yield an interest (Simple) of ₹ 560 at the rate of 14% per annum?

Solution: Here P = ₹ 2000, I = ₹ 560 R = 14%

$$I = P \times R \times T \text{ or } 560 = 2000 \times \frac{14}{100} \times T$$

$$\therefore\ T = \frac{560 \times 100}{2000 \times 14} = 2 \text{ years}$$

Thus, in 2 years, a sum of ₹ 2000 will yield an interest of ₹ 560 at 14% per annum.

Example 8.33: A certain sum of money at simple interest amounts to ₹ 1300 in 4 years and to ₹ 1525 in 7 years. Find the sum and rate percent.

Solution: Here $1300 = \frac{P \times R \times 4}{100} + P$ ...(i)

and $1525 = \frac{P \times R \times 7}{100} + P$ ...(ii)

Subtracting (i) from (ii) $225 = \frac{P \times R \times 3}{100}$ or $\frac{P \times R}{100} = 75$

Putting in (i) we get

$$1300 = 75 \times 4 + P \text{ or } P = ₹\ (1300 - 300) = ₹\ 1000$$

Again, we have $\frac{P \times R}{100} = 75$ or $R = \frac{75 \times 100}{P} = \frac{75 \times 100}{1000} = 7.5\%$

$\therefore$ Principal = ₹ 1000 and rate = 7.5%

Alternatively:

Amount after 4 years = ₹ 1300

Amount after 7 years = ₹ 1525

$\therefore$ Interest for 3 years = ₹ [1525 – 1300] = ₹ 225

$\therefore$ Interest for 1 year = ₹ $\frac{225}{3}$ = ₹ 75

$\therefore$ 1300 = P + Interest for 4 yrs = P + 4 × 75 or P = ₹ (1300 – 300) = ₹ 1000

$$R = \frac{75 \times 100}{1000 \times 1} = 7.5\%$$

Example 8.34: A certain sum of money doubles itself in 10 years. In how many years will it become $2\frac{1}{2}$ times at the same rate of simple interest.

Solution: Let P = ₹ 100, T = 10 yrs, A = ₹ 200, $\therefore$ I = ₹ 100

$$\therefore 100 = \frac{100 \times R \times 10}{100} \text{ or } R = 10\%$$

Now P = ₹ 100, R = 10% and A = ₹ 250 $\therefore$ I = ₹ 150

$$\therefore 150 = 100 \times \frac{10}{100} \times T \text{ or } T = 15 \text{ yrs}$$

Thus, in 15 yrs, the sum will become $2\frac{1}{2}$ times.

Example 8.35: Out of ₹ 70,000 to invest for one year, a man invests ₹ 30,000 at 4% and ₹ 20,000 at 3% per annum simple interest. At what rate percent, should he lend the remaining money, so that he gets 5% interest on the total amount he has?

Solution: Interest on total amount at 5% for one year

$$= ₹\ 70{,}000 \times \frac{5}{100} \times 1 = ₹\ 3500$$

Interest on ₹ 30,000 at 4% for 1 year $= ₹\ 30000 \times \frac{4}{100} \times 1$

$= ₹\ 1200$

Interest on ₹ 20,000 at 3% for 1 year $= ₹\ 20000 \times \frac{3}{100} \times 1$

$= ₹\ 600$

$\therefore$ Interest on remaining ₹ 20,000 for 1 yr = ₹ [3500 – 1200 – 600]

$= ₹\ 1700$

$$\therefore 1700 = 20000 \times \frac{R}{100} \times 1 \text{ or } R = \frac{1700 \times 100}{20000} = 8.5\%$$

$\therefore$ The remaining amount should be invested at 8.5% per annum.

CHECK YOUR PROGRESS 8.5

1. Rama borrowed ₹ 14000 from her friend at 8% per annum simple interest. She returned the money after 2 years. How much did she pay back altogether?
2. Ramesh deposited ₹ 15600 in a financial company, which pays simple interest at 8% per annum. Find the interest he will receive at the end of 3 years.
3. Naveen lent ₹ 25000 to his two friends. He gave ₹ 10,000 at 10% per annum to one of his friend and the remaining to other at 12% per annum. How much interest did he receive after 2 years.
4. Shalini deposited ₹ 29000 in a finance company for 3 years and received ₹ 38570 in all. What was the rate of simple interest per annum?
5. In how much time will simple interest on a sum of money be $\frac{2}{5}$ th of the sum, at the rate of 10% per annum.
6. At what rate of interest will simple interest be half the principal in 5 years.
7. A sum of money amounts to ₹ 1265 in 3 years and to ₹ 1430 in 6 years, at simple interest. Find the sum and the rate percent.
8. Out of ₹ 75000 to invest for one year, a man invested ₹ 30000 at 5% per annum and ₹ 24000 at 4% per annum. At what percent per annum, should he invest the remaining money to get 6% interest on the whole money.
9. A certain sum of money doubles itself in 8 years. In how much time will it become 4 times of itself at the same rate of interest?
10. In which case, is the interest earned more:

    (a) ₹ 5000 deposited for 5 years at 4% per annum, or

    (b) ₹ 4000 deposited for 6 years at 5% per annum?

8.5.4 Compound Interest

In the previous section, you have studied about simple interest. When the interest is calculated on the Principal for the entire period of loan, the interest is called simple interest and is given by

$$I = P \times R \times T$$

But if this interest is due (not paid) after the decided time period, then it becomes a part of the principal and so is added to the principal for the next time preiod, and the interest is calculated for the next time period on this new principal. Interest calculated, this way is called compound interest.

The time period after which the interest is added to the principal for the next time period is called the **Conversion Period.**

The conversion period may be one year, six months or three months and the interest is said to compounded, annually, semi-annually or quarterly, respectively. Let us take an example:

Example 8.36: Find the compound interest on a sum of Rs. 2000, for two years when the interest is compounded annually at 10% per annum.

Solution: Here P = ₹ 2000 and R = 10%

∴ Interest for the first conversion time period (i.e. first year)

$$= ₹\, 2000 \times \frac{10}{100} \times 1 = ₹\, 200$$

∴ Principal for the second year (or 2nd conversion period)

$$= ₹\, (2000 + 200) = ₹\, 2200$$

∴ Interest for the 2nd time period $= ₹\, 2200 \times \frac{10}{100} \times 1 = ₹\, 220$

∴ Amount payable at the end of two years = ₹ (2200 + 220)

$$= ₹\, 2420$$

∴ Total interest paid at the end of two years = ₹ (2420 – 2000)

$$= ₹\, 420$$

or [₹ (200 + 220) = ₹ 420]

∴ Compound interest = ₹ 420

Thus, for calculating the compound interest, the interest due after every coversion period is added to the principal and then interest is calculated for the next period.

8.5.4.1 Formula for Compound Interest

Let a sum P be borrowed for n years at the rate of r% per annum, then

$$\text{Interest for the first year} = P \times \frac{r}{100} \times 1 = \frac{Pr}{100}$$

Amount after one year $=$ Principal for 2nd year $= P + \frac{Pr}{100}$

$$= P\left(1 + \frac{r}{100}\right)$$

Interest for 2nd year $= P\left(1 + \frac{r}{100}\right) \times \frac{r}{100} \times 1 = \frac{Pr}{100}.\left(1 + \frac{r}{100}\right)$

Amount after 2 years $= P\left(1 + \frac{r}{100}\right) + \frac{Pr}{100}\left(1 + \frac{r}{100}\right) = P\left(1 + \frac{r}{100}\right)\left(1 + \frac{r}{100}\right)$

$$= P\left(1 + \frac{r}{100}\right)^2$$

Similarly, amount after 3 years $= P\left(1 + \frac{r}{100}\right)^3$ and so on.

Amount after n years $= P\left(1 + \frac{r}{100}\right)^n$

Thus, if A represents the amount and R represents r% or $\frac{r}{100}$, then

$$A = P(1 + R)^n = P\left(1 + \frac{r}{100}\right)^n$$

and compound interest $= A - P = P(1 + R)^n - P$

$$= P[(1 + R)^n - 1] \text{ or } \quad P\left[\left(1 + \frac{r}{100}\right)^n - 1\right]$$

Note: Simple interest and compound interest are equal for first year (first conversion period)

Example 8.37: Calculate the compound interest on ₹ 20,000 for 3 years at 5% per annum, when the interest is compounded annually.

Solution: Here P = ₹ 20,000, R = 5% and n = 3

$\therefore$ CI $= P[(1 + R)^n - 1]$

$$= ₹\ 20000\left[\left(1 + \frac{5}{100}\right)^3 - 1\right]$$

$$= ₹\left[\left(\frac{21}{20}\right)^3 - 1\right] = ₹\ 20000 \times \left[\frac{9261 - 8000}{8000}\right]$$

$$= ₹\ 3152.50$$

Example 8.38: Calculate the compound interest on ₹ 20,000 for $1\frac{1}{2}$ years at the rate of 10% per annum, when the interest is compounded semi-annually.

Solution: Here P = ₹ 20,000, R = 10% per annum

= 5% per half year

and $n = 1\frac{1}{2}$ yrs = 3 half years

$$\therefore \text{CI} = P[(1+R)^n - 1] = ₹\ 20{,}000\left[\left(1+\frac{5}{100}\right)^3 - 1\right]$$

$$= ₹\ 20{,}000 \times \left[\frac{9261}{8000} - 1\right] = ₹\ 3152.50$$

Example 8.39: Calculate the compound interest on ₹ 20,000 for 9 months at the rate of 4% per annum, when the interest is compounded quarterly.

Solution: Here P = ₹ 20,000, R = 4% per annum

= 1% per quarter of year

and n = 3/4 yrs = 3 quarters

$$\therefore \text{CI} = P[(1+R)^n - 1] = ₹\ 20{,}000\left[\left(1+\frac{1}{100}\right)^3 - 1\right]$$

$$= ₹\ 20{,}000 \times \left[\left(\frac{101}{100}\right)^3 - 1\right] = ₹\ \frac{20000 \times 30301}{100 \times 100 \times 100}$$

$$= ₹\ 606.02$$

Example 8.40: calculate the amount and compound interest on ₹ 12000 for $1\frac{1}{2}$ years at the rate of 10% per annum compounded annually.

Solution: Here P = ₹ 12000, R = 10% and $n = 1\frac{1}{2}$ years

Since interest is compounded, annually, so, amount at the end of 1 year is given by

$$A = P\left(1+\frac{R}{100}\right)^1 = ₹\ 12000\times\left(1+\frac{10}{100}\right)^1$$

$$= ₹\ 12000\times\frac{11}{10} = ₹\ 13200$$

$\therefore$ Principal for next 6 months = ₹ 13200

and Rate $R = \frac{10}{2}\% = 5\%$

$$\therefore\ A = ₹\ 13200\left(1+\frac{5}{100}\right)^1 = ₹\ 13200\times\frac{21}{20}$$

$$= ₹\ 13860$$

$\therefore$ Amount after $1\frac{1}{2}$ years = ₹ 13860

Compound interest = ₹ [13860 – 12000]

= ₹ 1860

Note: We can calculate the amount for $1\frac{1}{2}$ yrs as

$$A = ₹\ 12000\left(1+\frac{10}{100}\right)^1\left(1+\frac{5}{100}\right)^1$$

Example 8.41: At what rate percent per annum, will a sum of ₹ 15,625 become ₹ 17576 in three years, when the interest is compounded annually?

Solution: Here A = ₹ 17576, P = ₹ 15,625 and n = 3

Let R = r% per annum

$$\therefore 17576 = 15625\left(1+\frac{r}{100}\right)^3$$

$$\therefore \left(1+\frac{r}{100}\right)^3 = \frac{17576}{15625} = \left(\frac{26}{25}\right)^3$$

$$\therefore \left(1+\frac{r}{100}\right) = \frac{26}{25} \text{ or } \frac{r}{100} = \frac{26}{25} - 1 = \frac{1}{25}$$

or $r = \frac{100}{25} = 4$

$\therefore$ Rate = 4% per annum.

Example 8.42: In how much time will a sum of ₹ 8000 amount to ₹ 9261 at 10% per annum, compounded semi-annually?

Solution: Here A= ₹ 9261, P = ₹ 8000 and n = x semi yrs

R = 10% per annum = 5% semi annually

$$\therefore 9261 = 8000\left(1+\frac{5}{100}\right)^x$$

$$\text{or } \frac{9261}{8000} = \left(\frac{21}{20}\right)^x \text{ or } \left(\frac{21}{20}\right)^3 = \left(\frac{21}{20}\right)^x \quad \therefore x = 3$$

$\therefore$ Time = 3 half years = $1\frac{1}{2}$ years

Example 8.43: Find the difference between simple interest and compound interest for $1\frac{1}{2}$ years at 4% per annum, for a sum of ₹ 24000, when the interest is compounded semi-annually..

Solution: Here P = ₹ 24000, R = 4% per annum

$T = \frac{3}{2}$ years R = 2% per semi-annually

$n = 1\frac{1}{2}$ years = $\frac{3}{2}$ years = 3 semi years

Simple Interest = P × R × T $= ₹\ 24000 \times \frac{4}{100} \times \frac{3}{2}$

= ₹ 1440.

For compound interest, $A = P\left[\left(1+\frac{R}{100}\right)^n\right]$

$$A = ₹\ 24000\left[\left(1+\frac{2}{100}\right)^3\right]$$

$$A = ₹\ 24000\left[\left(\frac{51}{50}\right)^3\right] = ₹\ 24000\left[\frac{51}{50}\times\frac{51}{50}\times\frac{51}{50}\right]$$

$$= ₹\ \frac{24\times51\times51\times51}{125} = ₹\ 25468.99 \text{ or } ₹\ 25469$$

$$\therefore \text{ CI} = ₹\ [25469 - 24000] = ₹\ 1469$$

$$\text{Difference} = \text{CI} - \text{SI} \quad = ₹\ [1469 - 1440]$$

$$= ₹\ 29$$

Example 8.44: A sum of money is invested at compound interest for $1\frac{1}{2}$ year at 4% compounded annually. If the interests were compounded semi-annually, it would have fetched ₹ 20.40 more than in the previous case. Find the sum.

Solution: Let the sum be ₹ x.

Here $\quad R = 4\%$ annually, or 2% semi-annually

$$T = 1\frac{1}{2} \text{ yrs or 3 semi years}$$

In first case

$$A = ₹\ x\left[1+\frac{4}{100}\right]^1\left[1+\frac{2}{100}\right]^1$$

$$= ₹\ x\left(\frac{26}{25}\right)\left(\frac{51}{50}\right) = ₹\ \frac{1326\,x}{1250}$$

In 2nd case

$$A = ₹\ x\left(1+\frac{2}{100}\right)^3 = ₹\ x\left(\frac{51}{50}\right)^3$$

$$= ₹\ \frac{132651}{125000}$$

$$\therefore \text{ Difference} = ₹\ \left[\frac{132651}{125000}x - \frac{1326}{1250}x\right]$$

$$= ₹\ \frac{51x}{125000}$$

$$\therefore \frac{51x}{125000} = \frac{2040}{100} \quad \text{or} \quad x = ₹\ \frac{2040}{100} \times \frac{125000}{51} = ₹\ 5000$$

$$\therefore \text{Sum} = ₹\ 50{,}000$$

CHECK YOUR PROGRESS 8.6

1. Calculate the compound interest on ₹ 15625 for 3 years at 4% per annum, compounded annually.
2. Calculate the compound interest on ₹ 15625 for $1\frac{1}{2}$ years at 8% per annum, compounded semi-annually.
3. Calculate the compound interest on ₹ 16000 for 9 months at 20% per annum, compounded quarterly
4. Find the sum of money which will amount to ₹ 27783 in 3 years at 5% per annum, the interest being compounded annually.
5. Find the difference between simple interest and compound interest for 3 years at 10% per annum, when the interest is compounded annually on ₹ 30,000.
6. The difference between simple interest and compound interest for a certain sum of money at 8% per annum for $1\frac{1}{2}$ years, when the interest is compounded half-yearly is ₹ 228. Find the sum.
7. A sum of money is invested at compound interest for 9 months at 20% per annum, when the interest is compounded half yearly. If the interest were compounded quarterly, it would have fetched ₹ 210 more than in the previous case. Find the sum.
8. A sum of ₹ 15625 amounts to ₹ 17576 at 8% per annum, compounded semi-annually. Find the time.
9. Find the rate at which ₹ 4000 will give ₹ 630.50 as compound interest in 9 months, interest being compounded quarterly.
10. A sum of money becomes ₹ 17640 in two years and ₹ 18522 in 3 years at the same rate of interest, compounded annually. Find the sum and the rate of interest per annum.

8.5.5 Rate of Growth and Depreciation

In our daily life, we come across the terms like growth of population, plants, viruses etc and depreciation in the value of articles like machinery, crops, motor cycles etc.

The problems of growth and depreciation can be solved using the formula of compound interest derived in the previous section.

If V_o is the value of an article in the beginning and V_n is its value after 'n' conversion periods and the rate of growth/depreciation for the period be denoted by r%, then we can write

$$V_n = V_o\left(1+\frac{r}{100}\right)^n \text{ in case of growth, and}$$

$$V_n = V_o\left(1-\frac{r}{100}\right)^n \text{ in case of depreciation.}$$

If the rate of growth/depreciation varies for each conversion period, then

$$V_n = V_o\left(1+\frac{r_1}{100}\right)\left(1+\frac{r_2}{100}\right)\left(1+\frac{r_3}{100}\right)\text{.... for growth, and}$$

$$V_n = V_o\left(1-\frac{r_1}{100}\right)\left(1-\frac{r_2}{100}\right)\left(1-\frac{r_3}{100}\right)\text{.... for depreciation.}$$

Let us take some examples to illustrate the above concepts.

Example 8.45: The population of a city is 9765625. What will be its population after 3 years, if the rate of growth of population is 4% per year?

Solution: Here V_o = 9765625, r = 4% and n = 3

$$\therefore\ V_3 = 9765625\left[1+\frac{4}{100}\right]^3$$

$$= 9765625\times\left(\frac{26}{25}\right)^3$$

$$= 10985000.$$

Hence, the population of that city after 3 years will be = 10985000.

Example 8.46: The cost of a car was ₹ 3,50,000 in January 2005. If the rate of depreciation is 15% for the first year and 10% for the subsequent years, find its value after 3 years.

Solution: Here V_o = ₹ 3,50,000

r_1 = 15%, r_2 = 10% and r_3 = 10%

$$\therefore V_3 = V_o\left(1-\frac{r_1}{100}\right)\left(1-\frac{r_2}{100}\right)\left(1-\frac{r_3}{100}\right)$$

$$= ₹\ 350000\left(1-\frac{15}{100}\right)\left(1-\frac{10}{100}\right)\left(1-\frac{10}{100}\right)$$

$$= ₹\ 350000\times\frac{17}{20}\times\frac{9}{10}\times\frac{9}{10} = ₹\ 2,40,975/-$$

$\therefore$ The value of car after 3 years = ₹ 240975.

Example 8.47: A plant gains its height at the rate of 2% per month of what was its height in the beginning of the month. If its height was 1.2 m in the beginning of January 2008, find its height in the beginning of April 2008, correct upto 3 places of decimal.

Solution: Here $V_o = 1.2$ m, $r = 2\%$, $n = 3$

$$\therefore V_3 = V_o\left(1+\frac{r}{100}\right)^n$$

$$= 1.2\left(1+\frac{2}{100}\right)^3 = 1.2\left(\frac{51}{50}\right)^3 = 1.2734 \text{ m}$$

$= 1.273$ m

Hence, height of plant in the beginning of April = 1.273 m.

Example 8.48: The virus of a culture decreases at the rate of 5% per hour due to a medicine. If the virus count in the culture at 11.00 AM was 2.3×10^7, find the virus count at 1.00 PM on the same day.

Solution: $V_o = 2.3 \times 10^7$, $r = 5\%$, $n = 2$

$$V_2 = 2.3\times10^7\left(1-\frac{5}{100}\right)^2 = 2.3\times10^7\times(0.95)^2$$

$= 2.076 \times 10^7$

Hence, the virus count at 1.00 PM is 2.076×10^7.

CHECK YOUR PROGRESS 8.7

1. The population of a town is 281250. What will be its population after 3 years, if the rate of growth of population is 4% per year?
2. The cost of a car was ₹ 4,36,000 in January 2005. Its value depreciates at the rate of 15% in the first year and then at the rate of 10% in the subsequent years. Find the value of the car in January 2008.

3. The cost of machinery is ₹ 360000 today. In the first year the value depreciates by 12% and subsequently, the value depreciates by 8% each year. By how much, the value of machinery has depreciated at the end of 3 years?
4. The application of manure increases the output of a crop by 10% in the first year, 5% in the second year and 4% in the third year. If the production of crop in the year 2005 was 3.5 tons per hectare, find the production of crop per hectare in 2008.
5. The virus of a culture decreases at the rate of 4% per hour due to a medicine. If the virus count in the culture at 9.00 AM was 3.5×10^8, find the virus count at 11.00 AM on the same day.
6. Three years back, the population of a village was 50000. After that, in the first year, the rate of growth of population was 5%. In the second year, due to some epidemic, the population decreased by 10% and in the third year, the population growth rate was noticed as 4%. Find the population of the town now.

LET US SUM UP

- Percent means 'per hundred'.
- Percents can be written as fractions as well as decimals and vice-versa.
- To write a percent as a fraction, we drop the % sign and divide the number by 100.
- To write a fraction as a percent, we multiply the fraction by 100, simplify it and suffix the % sign.
- To determine the specific percent of a number or quantity, we change the percent to a fraction or a decimal and then multiply.
- When the selling price is more than the cost price of the goods, there is a profit (or gain).
- When the selling price is less than the cost price of the goods, there is a loss.

  Profit (Gain) = S.P. – C.P. ; Loss = C.P. – S.P.

  $$\text{Gain\%} = \frac{\text{Gain}}{\text{C.P.}} \times 100 \quad ; \quad \text{Loss\%} = \frac{\text{Loss}}{\text{C.P.}} \times 100$$

  $$\text{S.P.} = \frac{100 + \text{Gain\%}}{100} \times \text{C.P.} \quad ; \quad \text{S.P.} = \frac{100 - \text{Loss\%}}{100} \times \text{C.P.}$$

- The simple interest (I.) on a principal (P) at the rate of R% for a time T years, is calculated, using the formula

  $\text{I.} = \text{P} \times \text{R} \times \text{T}$

- Discount is a reduction in the list price of goods.
- Discount is always calculated on the marked price of the goods
- (Marked price – discount), gives the price, which a customer has to pay while buying an article.
- Two or more successive discounts are said to form a discount series.
- A discount series can be reduced to a single discount.
- Sales tax is charged on the sale price of goods.
- An instalment plan enables a person to buy costlier goods.
- In the case of compound interest

  Amount (A) = P $(1 + R)^n$, where P is the Principal, R = rate% and n = time.
- Compount interest is greater than simple interest, except for the first conversion period.
- If V_o is the value of an article in the beginning and V_n is its value after 'n' conversion periods and 'r' be the rate of growth/depreciation per period, then

$$V_n = V_o\left(1+\frac{r}{100}\right)^n \text{ in case of growth, and}$$

$$V_n = V_o\left(1-\frac{r}{100}\right)^n \text{ in case of depreciation.}$$

- If the rate of growth/depreciation varies for each conversion period, then

$$V_n = V_o\left(1+\frac{r_1}{100}\right)\left(1+\frac{r_2}{100}\right)\left(1+\frac{r_3}{100}\right)\text{.... for growth, and}$$

$$V_n = V_o\left(1-\frac{r_1}{100}\right)\left(1-\frac{r_2}{100}\right)\left(1-\frac{r_3}{100}\right)\text{.... for depreciation.}$$

TERMINAL EXERCISE

1. Write each of the following as a percent

   (a) $\frac{9}{20}$ (b) $\frac{7}{10}$ (c) 0.34 (d) 0.06

2. Write each of the following as a decimal:

   (a) 36% (b) 410% (c) 2% (d) 0.35%

3. Write each of the following as fraction:

   (a) 0.12% (b) 2.5% (c) 25.5% (d) 255%

4. Find each of the following:

   (a) 23% of 500 (b) 2.5% of 800 (c) 0.4% of 1000 (d) 115% of 400

5. What percent of 700 is 294?
6. By what percent is 60 more than 45?
7. What number increased by 10% of itself is 352?
8. Find the number whose 15% is 270.
9. What number decreased by 7% of itself is 16.74?
10. If three fourth of the students of a class wear glasses, what percent of the students of the class do not wear glasses?
11. There are 20 eggs in a fridge and 6 of them are brown. What percent of eggs are not brown?
12. 44% of the students of a class are girls. If the number of girls is 6 less than the number of boys, how many students are there in the class?
13. During an election, 70% of the population voted. If 70,000 people cast their votes, what is the population of the town?
14. A man donated 5% of his monthly income to a charity and deposited 12% of the rest in a Bank. If he has Rs. 11704 left with him, what is his monthly income?
15. Ratan stores has a sale of ₹ 12000 on Saturday, while Seema stores had a sale of ₹ 15000 on that day. Next day, they had respective sales of ₹ 15000 and ₹ 17500. Which store showed more improvement in Sales?
16. A candidate has to secure 45% marks in aggregate of three papers of 100 marks each to get through. He got 35% marks in the first paper and 50% marks in the second paper. At least how many marks should he get in third paper to pass the examination?
17. The price of sugar rises by 25%. By how much percent should a householder reduce his consumption of sugar, so as not to increase his expenditure on sugar?
18. By selling 90 ball pens for ₹ 160, a person loses 20%. How many ball pens should he sell for Rs. 96, so as to have a gain of 20%?
19. A vendor bought bananas at 6 for 5 rupees and sold them at 4 for 3 rupees. Find his gain or loss percent.
20. A man bought two consignments of eggs, first at ₹ 18 per dozen and an equal number at ₹ 20 per dozen. He sold the mixed egges at ₹ 23.75 per dozen. Find his gain percent.

21. A man sells an article at a gain of 10%. If he had bought it for 10% less and sold it for ₹ 10 more, he would have gained 25%. Find the cost price of the article.

22. A pair of socks is marked at ₹ 80 and is being offered at ₹ 64. Find the discount percent being offered.

23. A dealer buys a table listed at ₹ 1800 and gets a discount of 25%. He spends ₹ 150 on transportation and sells it at a profit of 10%. Find the selling price of the table.

24. A T.V. set was purchased by paying ₹ 18750. If the discount offered by the dealer was 25%, what was the marked price of the TV set?

25. A certain sum of money was deposited for 5 years. Simple interest at the rate of 12% was paid. Calculate the sum deposited if the simple interest received by the depositor is ₹ 1200.

26. Simple intrest on a sum of money is $\frac{1}{3}$rd of the sum itself and the number of years is thrice the rate percent. Find the rate of interest.

27. In what time will ₹ 2700 yield the same interest at 4% per annum as ₹ 2250 in 4 years at 3% per annum?

28. The difference between simple interest on a sum of money for 3 years and for 2 years at 10% per annum is ₹ 300. Find the sum.

29. Find the sum which when invested at 4% per annum for 3 years will becomes ₹ 70304, when the interest is compounded annually.

30. the difference between compound interest and simple interest at 10% per annum in 2 years (compounded annually) is ₹ 50. Find the sum.

31. A sum of money becomes ₹ 18522 in three years and ₹ 19448.10 in 4 years at the same rate of interest, compounded annually. Find the sum and the rate of interest per annum.

32. Find the sum of money which will amount to ₹ 26460 in six months at 20% per annum, when the interest is compounded quarterly.

33. At what rate percent per annum will a sum of ₹ 12000 amount to ₹ 15972 in three years, when the interest is compounded annually?

34. The price of a scooter depreciates at the rate of 20% in the first year, 15% in the second year and 10% afterwards, what will be the value of a scooter now costing ₹ 25000, after 3 years.

35. The population of a village was 20,000, two years ago. It increased by 10% during first year but decreased by 10% in the second year. Find the population at the end of 2 years.

ANSWERS TO CHECK YOUR PROGRESS

8.1

1. (a) 48% (b) 45% (c) $41\frac{2}{3}\%$ (d) 40% (e) 20%

   (f) 30% (g) 36% (h) 126% (i) 288% (j) 98.48%

2. (a) $\frac{53}{100}$ (b) $\frac{17}{20}$ (c) $\frac{27}{160}$ (d) $\frac{137}{4000}$ (e) $\frac{1}{16}$

   (f) $\frac{7}{10}$ (g) $\frac{63}{400}$ (h) $\frac{1}{40000}$ (i) $\frac{947}{2000}$ (j) $\frac{21}{4000}$

3. (a) 97% (b) 73.5% (c) 3% (d) 207% (e) 80%

   (f) 175% (g) 2.5% (h) 325.75% (i) 15.2% (j) 300.15%

4. (a) 0.72 (b) 0.41 (c) 0.04 (d) 1.25 (e) 0.09

   (f) 4.1 (g) 3.5 (h) 1.025 (i) 0.00025 (j) 0.1025

5. 50% 6. 90% 7. 6.25% 8. 47.5% 9. 30%

10. 5%

8.2

1. (a) 200 (b) 564
2. Rs. 2625 3. 175, 100, 125, 100 4. 25%
5. 56.25% 6. 500 7. 36 minutes
8. 6000 9. Rs. 40, Rs. 32 10. 4% decrease
11. B 12. 10.8%
13. Rs. 13200, Rs. 12000, Rs. 10000

8.3

1. $33\frac{1}{3}\%$ profit 2. 10% 3. 25% 4. 120

   5. Rs. 2576 6. 21 7. 4% loss 8. Rs. 1108.80

   9. 12% gain 10. 15% gain

8.4

1. Rs. 318.75 2. 20% 3. Rs. 15390 4. 25%
5. Rs. 724.50 6. 10% loss 7. Rs. 2185 8. 10.4%

8.5

1. Rs. 16240 2. Rs. 3744 3. Rs. 5600 4. 11%

5. 4 years	6. 10%	7. Rs. 1100, 5%	8. $9\frac{5}{7}\%$
9. 24 years	10. b		

8.6

1. Rs. 1951	2. Rs. 1951	3. Rs. 2522	4. Rs. 24000
5. Rs. 630	6. Rs. 46875	7. Rs. 80000	8. $1\frac{1}{2}$ years
9. 20%	10. Rs. 1600, 5%		

8.7

1. 316368	2. Rs. 300186	3. Rs. 291456
4. 4.2042 tons/hectare	5. 3.2256×10^8	6. 49140

ANSWERS TO TERMINAL EXERCISE

1. (a) 45%	(b) 70%	(c) 34%	(d) 6%
2. (a) 0.36	(b) 4.10	(c) 0.02	(d) 0.0035
3. (a) $\frac{3}{2500}$	(b) $\frac{1}{40}$	(c) $\frac{51}{200}$	(d) $\frac{51}{20}$
4. (a) 115	(b) 20	(c) 4	(d) 460
5. 42%	6. 25%	7. 320	8. 1800
9. 18	10. 25%	11. 70%	12. 50
13. 1 Lakh	14. Rs. 14000	15. Ratan Stores	16. 50
17. 20%	18. 36	19. 60% gain	20. 25%
21. Rs. 400	22. 20%	23. Rs. 1650	24. Rs. 25000
25. Rs. 2000	26. $3\frac{1}{3}\%$	27. $2\frac{1}{2}$ years	28. Rs. 3000
29. Rs. 62500	30. Rs. 5000	31. Rs. 16000, 5%	32. Rs. 24000
33. 10%	34. Rs. 13500	35. 19800	

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9

INSTALMENT BUYING

You must have seen advertisements like, “Pay just ₹ 500 and take home a color TV, rest in easy instalments”, or “buy a car of your choice by paying ₹ 50,000 and the balance in easy instalments”. Such plans attract customers, specially the common man, who could not buy some costly articles like car, scooter, fridge, colour TV, etc. due to cash constraints. Under these plans, a fixed amount is paid at the time of purchase and the rest of the amount is to be paid in instalments, which may be monthly, quarterly, half yearly or yearly, as per the agreement signed between the customer and the seller.

Instalment purchase scheme, thus, enables a person to buy costly goods, on convenient terms of payment. Under this scheme, the customer, after making a partial payment in the beginning, takes away the article for use after signing the agreement to pay the balance amount in instalments. Such a scheme also encourages the buyer to save at regular intervals, so as to pay the instalments.

In this lesson, we shall study different types of instalment plans and shall find out how much easy they are, by calculating the interest charged under these plans.

OBJECTIVES

After studying this lesson, you will be able to

- *explain the advantages/disadvantages of buying a commodity under instalment plan;*
- *determine the amount of each instalment, when goods are purchased under instalment plan at a given rate of interest (simple interest);*
- *determine the rate of interest when the amount of each (equal) instalment and the number of instalments is given;*
- *determine the amount of each instalment under instalment plan when compound interest is charged yearly, half yearly or quarterly;*
- *solve problems pertaining to instalment plan.*

EXPECTED BACKGROUND KNOWLEDGE

- Simple interest and compound interest.
- Calculation of interest when the interest is calculated yearly, half yearly, quarterly or monthly..

9.1 INSTALMENT BUYING SCHEME-SOME DEFINITIONS

Cash Price: The cash price of an article is the amount which a customer has to pay in full for the article at the time of purchase.

Cash Down Payment: The amount to be paid (in cash) under an instalment plan at the time of purchase of a commodity, is called the **cash down payment**. It is the partial payment made by the customer at the time of signing the agreement and taking away the article for use.

Instalments: It is the amount which is paid by the customer at regular intervals towards the remaining part of the selling price of the article.

Interest under the Instalment Plan: In an instalment plan only part payment of the total cost is paid by the customer at the time of purchase. The remaining part of cost is paid on subsequent dates; and therefore the seller charges some extra amount for deferred payments. This extra amount is actually the interest charged on the amount of money which the customer ows to the seller at different times of payment of instalments.

9.2 TO FIND THE INTEREST IN AN INSTALMENT PLAN

Let us solve a few examples to illustrate the process.

Example 9.1: A Television set is sold for ₹ 20000 cash or for ₹ 6000 as cash down payment followed by ₹ 16800 after six months. Find the rate of interest charged under the instalment plan.

Solution: The cash price of the television = ₹ 20000

Cash down payment = ₹ 6000

Balance to be paid = ₹ 14000

$\therefore$ The present value of Rs. 16800 to be paid after 6 months = Rs. 14000

If the rate of interest per annum under instalment plan is r%, then

$$14000+14000\times\frac{r}{100}\times\frac{6}{12}=16800$$

or $\frac{7r}{10}=28$ i.e., r = 40, i.e. rate = 40%

Example 9.2: A table fan is sold for ₹ 450 cash or ₹ 210 cash down payment followed by two monthly instalments of ₹ 125 each. Find the rate of interest charged under the instalment plan.

Solution: Cash price of the table fan = ₹ 450

Cash down payment = ₹ 210

Balance to be paid = ₹ (450 – 210) = ₹ 240

Let the rate of interest charged under instalment plan be r% p.a. then

₹ 240 at the end of two months will become $= ₹\left(240 + 240 \times \frac{r}{100} \times \frac{2}{12}\right)$

$$= ₹\left(240 + \frac{2r}{5}\right) \quad \text{...(i)}$$

₹ 125 paid after 1 month will amount to (after another 1 month)

$$= ₹\ 125 + 125 \times \frac{r}{100} \times \frac{1}{12} = Rs.\left(125 + \frac{5r}{48}\right) \quad \text{...(ii)}$$

Amount for ₹ 125 paid after two months = ₹ 125 ...(iii)

$$\therefore 240 + \frac{2r}{5} = 125 + \frac{5r}{48} + 125 \text{ i.e., } \left(\frac{2}{5} - \frac{5}{48}\right) r = 10$$

$$\Rightarrow r = \frac{2400}{71} = 33.8 \text{ (approx)}$$

Hence, rate of interest = 33.8%

Alternative method:

Cash price of the fan = ₹ 450

Cash down payment = ₹ 210

Payment in 2 instalments = ₹ (125 × 2) = ₹ 250

Total amount paid under instalment plan = ₹ (210 + 250)

= ₹ 460

∴ Interest paid = ₹ (460 – 450) = ₹ 10

The Principal for the first month = ₹ (450 – 210) = ₹ 240

Principal for the 2nd month = ₹ (240 – 125) = ₹ 115

$\therefore$ Total Principal (for 1 month) = ₹ (240 +115) = ₹ 355

Thus we have

$$355 \times \frac{r}{100} \times \frac{1}{12} = 10, \text{ or, } r = \frac{10 \times 100 \times 12}{355}$$

$$= \frac{2400}{71} \approx 33.8$$

Hence, rate of interest = 33.8% p.a.

Example 9.3: A microwave oven is available for ₹ 9600 cash or for ₹ 4000 cash down payment and 3 monthly instalments of ₹ 2000 each. Find the rate of interest charged under the instalment plan.

Solution: Cash price of microwave oven = ₹ 9600

Cash down payment = ₹ 4000

Payment in 3 instalments = ₹ (3 × 2000) = ₹ 6000

Total amount paid under instalment plan = ₹ (4000 + 6000)

= ₹ 10000

$\therefore$ Interest paid = ₹ (10000 – 9600) = ₹ 400

Principal for 1st month = ₹ (9600 – 4000) = ₹ 5600

Principal for 2nd month = ₹ (5600 – 2000) = ₹ 3600

Principal for 3rd month = ₹ (3600 – 2000) = ₹ 1600

$\therefore$ Total Principal (for 1 month) = ₹ (5600 + 3600 + 1600)

= ₹ 10800

Thus, we have

$$10800 \times \frac{r}{100} \times \frac{1}{12} = 400 \Rightarrow 9r = 400 \text{ or } r = \frac{400}{9} \approx 44.4\%$$

So, rate of interest charged = 44.4%

Example 9.4: A computer is sold for ₹ 30,000 cash or ₹ 18000 cash down payment and 6 monthly instalments of ₹ 2150 each. Find the rate of interest charged under the instalment plan.

Solution: Cash price of the computer = ₹ 30000

Cash down payment = ₹ 18000

Payment in 6 instalments = ₹ (6 × 2150) = ₹ 12900

$\therefore$ Total amount paid under instalment plan = ₹ (18000 + 12900)

= ₹ 30900

$\therefore$ Interest paid = ₹ (30900 – 30000) = ₹ 900

Principal for 1st month = ₹ (30000 – 18000) = ₹ 12000

Principal for 2nd month = ₹ (12000 – 2150) = ₹ 9850

Principal for 3rd month = ₹ (9850 – 2150) = ₹ 7700

Principal for 4th month = ₹ (7700 – 2150) = ₹ 5550

Principal for 5th month = ₹ (5550 – 2150) = ₹ 3400

Principal for 6th month = ₹ (3400 – 2150) = ₹ 1250

$\therefore$ Total Principal for one month = ₹ (12000 + 9850 + 7700 + 5550 + 3400 + 1250)

= ₹ 39750

$\therefore$ We have

$$39750\times\frac{r}{100}\times\frac{1}{12}=900 \Rightarrow r=\frac{900\times12\times100}{39750}=\frac{1440}{53}$$

$$= 27.17\%$$

Thus, the rate of interest = 27.17% per annum.

Note: In Examples 2 to 4, observe that the Principal for the last month is less than the amount of the instalment. If interest is added to the last Principal, the sum will be equal to the amount of monthly instalment.

CHECK YOUR PROGRESS 9.1

1. A table is sold for ₹ 2000 cash or for ₹ 600 as cash down payment, followed by ₹ 1500 paid after 2 months. Find the rate of interest charged under the instalment plan.
2. A cycle is sold for ₹ 2700 cash or ₹ 600 as cash down payment, followed by 3 monthly instalments of ₹ 750 each. Find the rate of interest charged under the instalment plan.
3. A T.V. set is available for ₹ 21000 cash or for ₹ 4000 cash down payment and 6 equal monthly instalments of ₹ 3000 each. Calculate the rate of interest charged under the instalment plan.
4. Anil purchased a computer monitor priced at ₹ 6800 cash, under the instalment plan by making a cash down payment of ₹ 2000 and 5 monthly instalments of ₹ 1000 each. Find the rate of interest charged under the instalment plan.
5. A scooter can be purchased for ₹ 28000 cash or for ₹ 7400 as cash down payment followed by 4 equal monthly instalments of ₹ 5200 each. Find the rate of interest charged under instalment plan.

6. An air conditioner is sold for ₹ 20,000 cash or ₹ 12000 cash down payment followed by 4 monthly instalments of ₹ 2200 each. Find the rate of interest under the instalment plan correct upto one decimal place.
7. An article is available for ₹ 25000 cash or 20% cash down payment followed by 6 monthly instalments of ₹ 3750 each. Calculate the rate of interest charged under the instalment plan.

9.3 TO FIND THE AMOUNT OF INSTALMENT

Now, let us think the problem with the shopkeeper's angle. A shopkeeper purchases an article at some price and wants to offer an instalment plan to his customers, as he knows that more items can be sold in this way. Now he wishes to charge interest at a particular rate and wants to decide the cash down payment, the amount of equal instalments and the number of instalments.

Let us take some examples to illustrate the process.

Example 9.5: A ceiling fan is marked at ₹ 1940 cash or for ₹ 420 cash down payment followed by three equal monthly instalments. If the rate of interest charged under the instalment plan is 16% per annum, find the monthly instalment.

Solution: Cash price of ceiling fan = ₹ 1940

Cash down payment = ₹ 420

Let each instalment = ₹ x

∴ Amount paid in instalment plan = ₹ [420 + 3x]

∴ Interest paid = ₹ (420 + 3x – 1940) = ₹ (3x – 1520)

The buyer owes to the seller for first month = ₹ 1520

The buyer owes to the seller for 2nd month = ₹ (1520 – x)

The buyer owes to the seller for 3rd month = ₹ (1520 – 2x)

∴ Totol principal for one month = ₹ [4560 – 3x]

Rate of interest = 16%

$$\therefore (3x - 1520) = (4560 - 3x)\frac{16}{100}\cdot\frac{1}{12}$$

$$25(3x - 1520) = (1520 - x)$$

i.e., $76x = 39520$

or $x = 520$

So, the amount of each instalment = ₹ 520

Example 9.6: A computer is available for ₹ 34000 cash or ₹ 20000 cash down payment together with 5 equal monthly instalments. If the rate of interest charged under the instalment plan is 30% per annum, calculate the amount of each instalment.

Solution: Cash price = ₹ 34000

Cash down payment = ₹ 20000

Balance to be paid in 5 equal instalments = ₹ 14000

Let each instalment be ₹ x

So, interest charged under instalment plan = ₹ (5x – 14000)

The buyer owes to the seller for

1st month	2nd month	3rd month	4th month	5th month
₹ 14000	₹ (14000 –x)	₹ (14000 –2x)	₹ (14000 –3x)	₹ (14000 –4x)

Therefore, total principal for one month = ₹ [70000 – 10x]

$$\text{So, } (5x - 14000) = (70000 - 10x) \times \frac{30}{100} \times \frac{1}{12}$$

40 (5x – 14000) = 10(7000 – x)

20x – 56000 = 7000 – x

or 21x = 63000

or x = 3000

Thus, the amount of each instalment = ₹ 3000

Example 9.7: The cost of a washing machine is ₹ 12000. The company asks for ₹ 5200 in advance and the rest to be paid in equal monthly instalments. The rate of interest to be charged is 12% per annum. If a customer can pay ₹ 1400 each month, then how many instalments he will have to pay?

Solution: Let number of instalments be ‘n’

Cash price of washing machine = ₹ 12000

Price under instalment plan = ₹ (5200 + 1400n)

∴ Interest charged = ₹ (5200 + 1400n – 12000)

= ₹ (1400n – 6800)

Principal owed each month is

First month = ₹ 6800

2nd month = ₹ 5400

3rd month = ₹ 4000

4th month = ₹ 2600

5th month = ₹ 1200

6th month = nil

Total for one month = ₹ 20000

So, $$20000 \times \frac{12}{100} \times \frac{1}{12} = (1400n - 6800)$$

$1400\,n = 7000$ i.e. n = 5

Thus, the number of instalments = 5

CHECK YOUR PROGRESS 9.2

1. A scooter is available for ₹ 30000 cash or for ₹ 15000 cash down payment and 4 equal monthly instalments. If the rate of interest charged under the instalment plan is $33\frac{1}{3}$%, find the amount of each instalment.

2. A microwave oven is available for ₹ 9600 cash or for ₹ 4000 cash down payment and 3 equal monthly instalments. If the rate of interest charged is $22\frac{2}{9}$% per annum, find the amount of each instalment.

3. An article is sold for ₹ 5000 cash or for ₹1500 cash down payment followed by 5 equal monthly instalments. If the rate of interest charged is 18% p.a., compute the amount of each monthly instalment.

4. An article is sold for ₹ 500 cash or ₹ 150 cash down payment followed by 5 equal monthly instalments. If the rate of interest charged is 18% per annum, compute the monthly instalment.

9.4 TO FIND CASH PRICE

Let us now take problems where we are to find the cash price of an article when in the instalment scheme, amount of each equal instalment, the rate of interest, the number of instalments and the amount of cashdown payment, are given.

Example 9.8: A bicycle is sold for ₹ 500 cash down payment and ₹ 610 after one month. If the rate of interest being charged is 20% p.a., find the cash price of the bicycle.

Solution: Cash down payment = ₹ 500

Amount of instalment paid after one month = ₹ 610

Rate of interest = 20%

Thus we have to find present value (i.e. Principal) of Rs. 610 paid after one month.

So, $$610 = \left[(\text{Principal}) \times \frac{20}{100} \times \frac{1}{12} + \text{Principal}\right]$$

$$\Rightarrow 610 = \text{Principal}\left(1 + \frac{20}{1200}\right) \text{ or Principal} = ₹\ \frac{610 \times 1200}{1220}$$

$$= ₹\ 600$$

$\therefore$ The cash price of bicycle = ₹ (500 + 600) = ₹ 1100

Example 9.9: A camera is sold for ₹ 2500 as cash down payment and ₹ 2100 after 3 months. If the rate of interest charged is 20% p.a., find the cash price of the camera.

Solution: Cash down payment = ₹ 2500

Instalment paid after 3 months = ₹ 2100

Rate of interest = 20% p.a.

So, Principal amount for ₹ 2100

$$= ₹\ \frac{2100 \times 100}{100 + 20 \times \frac{3}{12}} = ₹\ \frac{2100 \times 1200}{1260}$$

$$= ₹\ 2000$$

Therefore, cash price = ₹ (2500 + 2000) = ₹ 4500

Alternative Method:

Let cash price be ₹ x.

Cash down payment = ₹ 2500

Instalment paid = ₹ 2100

$\therefore$ Interest = ₹ (4600 – x)

Principal for the instalment = ₹ (x – 2500)

$$\therefore (4600 - x) = (x - 2500) \times \frac{3}{12} \times \frac{20}{100} = \frac{x - 2500}{20}$$

20(4600 – x) = x – 2500

or 21x = 92000 + 2500

or $\quad 21x = 94500$

or $\quad x = 4500$

Hence, cash price = ₹ 4500

Example 9.10: A mixi was purchased by paying ₹ 360 as cash down payment followed by three equal monthly instalments of ₹ 390 each. If the rate of interest charged under instalment plan is 16% p.a., find the cash price of the mixi.

Solution: Let the cash price of the mixi be ₹ x

Cash down payment = ₹ 360

Amount paid in 3 instalments = ₹ (3×390) = ₹ 1170

Total paid = ₹ (360 + 1170) = ₹ 1530

$\therefore$ Interest = ₹ $(1530 - x)$

Principal for 1st month = ₹ $(x - 360)$

Principal for 2nd month = ₹ $(x - 360 - 390)$ = ₹ $(x - 750)$

Principal for 3rd month = ₹ $(x - 750 - 390)$ = ₹ $(x - 1140)$

Total principal for one month = ₹ $[x - 360 + x - 750 + x - 1140]$

= ₹ $[3x - 2250]$

So, $$(1530 - x) = (3x - 2250) \times \frac{1}{12} \times \frac{16}{100} = \frac{(x - 750)}{25}$$

$25(1530 - x) = x - 750$

or $\quad 26x = 38250 + 750 = 39000$

or $$x = \frac{39000}{26} = 1500$$

Thus, the cash price of mixi = ₹ 1500

CHECK YOUR PROGRESS 9.3

1. A table was purchased by paying a cash down payment of ₹ 750 followed by ₹ 436 after a period of 6 months. If the rate of interest charged is 18% p.a., what is the cash price of the table?
2. A refrigerator was purchased for a cashdown payment of ₹ 7000 followed by a sum of ₹ 3180 after 3 months. If the rate of interest charged is 24% p.a., find the cash price of the refrigerator.

3. A cooking range is available for ₹ 520 cash down payment followed by 4 equal monthly instalments of ₹ 520 each. If the rate of interest charged is 25% per annum, find the cash price of the cooking range.
4. A ceiling fan was purchased for ₹ 210 as cash down payment followed by three equal instalments of ₹ 260 each. If the rate of interest charged under the instalment plan is 16% p.a., then find the cash price of the ceiling fan.
5. An electrical oven was purchased for ₹ 1500 cash down payment, followed by five equal monthly instalments of ₹ 440 each. If the rate of interest charged per annum under the instalment plan is 24%, find the cash price of the oven

9.5 PROBLEMS INVOLVING COMPOUND INTEREST

In instalment buying which involved monthly instalments with the total time period being less than a year, simple interest was used.

Sometimes the individuals take long-term loans, for purposes like, buying a house, a car or setting up a factory etc. In that case, the instalments are to be paid annually for a long period and therefore involves the use of compound interest. Even in instalment buying for a period less than a year, sometimes the seller charges compound interest when the instalments are semi annually or quarterly.

Now, we shall take some problems involving compound interest.

Example 9.11: A refrigerator is available for ₹ 12000 cash or ₹ 3600 cash down payment along with 2 equal half yearly instalments. If the dealer charges an interest of 20% p.a. compounded semi-annually, under the instalment plan, find the amount of each instalment.

Solution: Cash price of refrigerator = ₹ 12000

Cash down payment = ₹ 3600

Balance = ₹ 8400

Rate of interest = 20% p.a. or 10% semi-annually

Let each monthly instalment be ₹ x, then we shall find the present value (or the Principal) for each instalment.

Let P_1, P_2 be the present values of first, 2nd conversion period respectively.

$$\therefore \quad x = P_1\left(1+\frac{10}{100}\right)^1 \text{ and } x = P_2\left(1+\frac{10}{100}\right)^2$$

$$\text{Therefore, } P_1 = \frac{10}{11}x \text{ and } P_2 = \left(\frac{10}{11}\right)^2 x$$

$$\text{Thus, we have, } \frac{10}{11}x + \frac{100}{121}x = 8400$$

or $$x = \frac{8400 \times 121}{210} = 4840$$

Thus, the amount of each instalment = ₹ 4840.

Example 9.12: A washing machine was available for ₹ 15000 cash but was purchased under an instalment plan after paying ₹ 2250 as cash down payment followed by two equal half yearly instalments. If interest charged was 8% per annum compounded semi-annually, find the value of each instalment.

Solution: Cash price of the washing machine = ₹ 15000

Cash down payment = ₹ 2250

Balance to be paid = ₹ [15000 – 2250] = ₹ 12750

Rate of interest = 8% p.a. = 4% semi-annually

Let each instalment be ₹ x (semi-annually) and

P_1, P_2 be the present values respectively of the two instalments, then

$$\therefore\ x = P_1\left(1+\frac{4}{100}\right)^1 \text{ and } x = P_2\left(1+\frac{4}{100}\right)^2$$

This gives $P_1 = \frac{25}{26}x$ and $P_2 = \left(\frac{25}{26}\right)^2 x$

Hence, $12750 = \frac{25}{26}x + \left(\frac{25}{26}\right)^2 x = \frac{25}{26}x\left(1+\frac{25}{26}\right) = \frac{25}{26}\cdot\frac{51}{26}x$

$$\Rightarrow x = 12750 \times \frac{26}{25} \times \frac{26}{51} = 6760$$

Thus, each instalment = ₹ 6760.

Example 9.13: A juicer is available for ₹ 3500 cash but was sold under instalment plan where the purchaser agreed to pay ₹ 1500 cash down and 3 equal quarterly instalments. If the dealer charges interest at 12% p.a. compounded quarterly, find the amount of each instalment to the nearest rupee.

Solution: Cash price of the juicer = ₹ 3500

Cash down payment = ₹ 1500

Balance to be paid = ₹ (3500 – 1500) = ₹ 2000

Rate of interest = 12% p.a. = $\frac{12}{4}$ = 3% quarterly

Let the amount of each instalment be Rs. x and P_1, P_2, P_3 respectively be their present values, then

$$x = P_1\left(1+\frac{3}{100}\right), \quad x = P_2\left(1+\frac{3}{100}\right)^2 \quad \text{and} \quad x = P_3\left(1+\frac{3}{100}\right)^3$$

$$P_1 = \frac{100}{103}x, \quad P_2 = \left(\frac{100}{103}\right)^2 x \quad \text{and} \quad P_3 = \left(\frac{100}{103}\right)^3 x$$

$$\frac{100}{103}x + \left(\frac{100}{103}\right)^2 x + \left(\frac{100}{103}\right)^3 x = 2000 \Rightarrow \frac{100}{103}x\left[1+\frac{100}{103}+\left(\frac{100}{103}\right)^2\right] = 2000$$

$$x = 2000 \times \frac{103}{100} \times \frac{(103)^2}{30909} = ₹\ 707$$

$\therefore$ Each instalment = ₹ 707

Example 9.14: A television set is sold for ₹ 7110 cash down payment along with 2 equal monthly instalments of ₹ 5581.50 each. If the dealer charges interest at 20% p.a. compounded monthly under the instalment plan, find the cash price of the television set.

Solution: Cash down payment = ₹ 7110

Amount of each monthly instalment = ₹ 5581.50 = ₹ $\frac{11163}{2}$

Rate of interest = 20% p.a. = $\frac{20}{12}$ monthly

Let P_1, P_2 be the Principals for 1st and 2nd instalment respectively

$$\frac{11163}{2} = P_1\left(1+\frac{20}{1200}\right) \quad \text{and} \quad \frac{11163}{2} = P_2\left(1+\frac{20}{1200}\right)^2$$

This gives $P_1 = \frac{11163}{2} \times \frac{60}{61} = \text{Rs.}5490$ and $P_2 = \frac{11163}{2} \times \frac{60}{61} \times \frac{60}{61} = \text{Rs.}5400$

Thus, cash Price = ₹ [7110 + 5490 + 5400] = ₹ 18000

Example 9.15: A dealer profferes a micro-oven for ₹ 5800 cash. A customer agrees to pay ₹ 1800 cash down and 3 equal annual instalments. If the dealer charges interest at 12% p.a. compounded annually, what is the amount of each instalment.

Solution: Cash price of the micro-oven = ₹ 5800

Cash down payment = ₹ 1800

Balance to be paid = ₹ 4000

Rate of interest = 12% p.a. compounded annually

$\therefore$ Let Rs. x be the amount of each instalment and P_1, P_2, P_3 be the principals for each instalment respectively.

$$\therefore\ x = P_1\left(1+\frac{12}{100}\right),\quad x = P_2\left(1+\frac{12}{100}\right)^2 \quad \text{and} \quad x = P_3\left(1+\frac{12}{100}\right)^3$$

$$\Rightarrow\ P_1 = \frac{25}{28}x,\ P_2 = \left(\frac{25}{28}\right)^2 x \quad \text{and} \quad P_3 = \left(\frac{25}{28}\right)^3 x$$

$$\therefore\ \frac{25}{28}x + \left(\frac{25}{28}\right)^2 x + \left(\frac{25}{28}\right)^3 x = 4000$$

or $$\frac{25}{28}x\left(1+\frac{25}{28}+\frac{625}{784}\right) = 4000$$

or $$x = 4000\times\frac{28}{25}\times\frac{784}{2109} = ₹\ 1665.40$$

Hence each instalment = ₹ 1665.40

Example 9.16: A flat is available for ₹ 1600000 cash or ₹ 585500 cash down payment and three equal half yearly instalments. If the interest charged is 16% per annum compounded half yearly, calculate the value of each instalment. Find also the total interest charged.

Solution: Cash price of the flat = ₹ 1600000

Cash down payment = ₹ 585500

Balance to be paid = ₹ 1014500

Rate of interest = 16% per annum = 8% semi annually

Let the amount of each instalment be ₹ x and Let P_1, P_2 and P_3 be the Principals for each instalment respectively.

$$\text{So,}\ x = P_1\left(1+\frac{8}{100}\right) \text{ or } x = P_1\left(\frac{27}{25}\right) \text{ or } P_1 = x\left(\frac{25}{27}\right)$$

$$\text{Similarly, } P_2 = x\left(\frac{25}{27}\right)^2 \quad \text{and} \quad P_3 = x\left(\frac{25}{27}\right)^3$$

$$\therefore\ P_1 + P_2 + P_3 = 1014500$$

$$x\left(\frac{25}{27}\right)+x\left(\frac{25}{27}\right)^2+x\left(\frac{25}{27}\right)^3=1014500$$

$$x\left(\frac{25}{27}\right)\left[1+\frac{25}{27}+\left(\frac{25}{27}\right)^2\right]=1014500$$

$$x.\frac{25}{27}.\frac{2029}{729}=1014500$$

$$\text{x} = \frac{1014500\times 27\times 729}{25\times 2029}$$

$$= ₹\ 393660$$

Interest paid = ₹ [393660 × 3 – 1014500]

= ₹ [1180980 – 1014500]

= ₹ 166480.

CHECK YOUR PROGRESS 9.4

1. A bicycle is available for ₹ 1661 cash or by paying ₹ 400 cash down and balance in three equal half yearly instalments. If the interest charged is 10% per annum compounded semi-annually, find the instalment.

2. A washing machine is available for ₹ 15000 cash or ₹ 2000 cash down with two equal half yearly instalments. If the rate of interest charged is 16% per annum compounded half yearly, find the instalment.

3. Kamal purchased a computer in instalment plan by paying ₹ 5612.50 cash down followed by three equal quarterly instalments of ₹ 8788 each. If the rate of interest charged was 16% per annum, compounded quarterly, find the cash price of the computer. Also find the total interest charged.

4. A car was available for ₹ 70000 cash or by paying ₹ 21200 cash down along with three equal annual instalments. If the dealer charges interest of 25% per annum, compounded annually, find the amount of each instalment.

5. A microwave oven was purchased by paying a cash down payment of ₹ 2800 along with 2 equal annual instalments of ₹ 2420 each. If the rate of interest charged under the instalment plan was 10% p.a. compounded annually, find the cash price of the article.

LET US SUM UP

- Under an instalment scheme, the customer, after making a partial payment in the beginning takes away the article for use, after signing the agreement to pay the balance amount in instalments.
- Under instalment plan, the buyer pays some extra amount, which is **interest** on the deferred payments.
- Instalment scheme encourages the buyer to save at regular intervals, so as to pay the instalments.
- The price at which the article is available, if full payment is made in cash, is called the **cash price** of the article.
- The partial payment made at the time of purchase under instalment plan is called **Cash down payment.**
- The payments, which the buyer has to make at regular intervals, are called instalments.

TERMINAL EXERCISE

1. A sewing machine is available for ₹ 2600 cash payment or under an instalment plan for ₹ 1000 cash down payment and 3 equal monthly instalments of ₹ 550 each. Find the rate of interest charged under the instalment plan.
2. Anil purchased a typewriter priced at ₹ 8000 cash payment under the instalment plan by making a cashdown payment of ₹ 3200 and 5 equal monthly instalments of ₹ 1000 each. Find the rate of interest charged under the instalment plan.
3. A table is sold for ₹ 2000 cash or ₹ 500 as cash payment followed by 4 equal monthly instalments of ₹ 400 each. Find the rate of interest charged under the instalment plan.
4. A T.V. set has a cash price of ₹ 7500 or ₹ 2000 as cash down payment followed by 6 monthly instalments of ₹ 1000 each. Find the rate of interest charged under instalment plan.
5. An article is available for ₹ 7000 cash or for ₹ 1900 cash down payment and six equal monthly instalments. If the rate of interest charged is $2\frac{1}{2}$% per month, determine each instalment.
6. An article is sold for ₹ 1000 cash or Rs. 650 cash down payment followed by 5 equal monthly instalments. If the rate of interest charged is 18% per annum, compute the monthly instalment.

7. The selling price of a washing machine is ₹ 14000. The company asks for ₹ 7200 in advance and the rest to be paid in equal monthly instalments of ₹ 1400 each. If the rate of interest is 12% per annum, find the number of instalments.

8. A scooter is available for ₹ 30000 cash or for ₹ 15000 cash down payment and 4 equal monthly instalments. If the rate of interest charged under the instalment plan is $33\frac{1}{3}$%, find the amount of each instalment.

9. A plot of land is available for ₹ 200000 cash or ₹ 100000 cash down payment and 5 monthly instalments of ₹ 21000 each. Find the rate of interest charged under the instalment plan.

10. A steel almirah is marked for ₹ 3575 cash or ₹ 1600 as cash down payment and ₹ 420 per month for 5 months. Find the rate of interest under the instalment plan.

11. A watch is sold for ₹ 1000 cash or for ₹ 300 cash down payment followed by 5 equal monthly instalments. If the rate of interest charged is 18% p.a., compute the monthly instalment.

12. A computer is available for ₹ 34000 cash or ₹ 20000 cash down payment, together with 5 equal monthly instalments. If the rate of interest charged under instalment plan is 30% per annum, calculate the amount of each instalment.

13. Rita purchased a washing machine for ₹ 4000 cash down payment and 4 equal monthly instalments. The washing machine was also available for ₹ 15000 cash payment. If the rate of interest charged under the instalment plan is 18% per annum, find the amount of each instalment.

14. A ceiling fan is marked at ₹ 970 cash or ₹ 210 cash down payment followed by three equal monthly instalments. If the rate of interest charged under the instalment plan is 16% p.a., find the monthly instalment.

15. A watch is available for ₹ 970 cash or for ₹ 350 as cash down payment followed by 3 equal monthly instalments. If the rate of interest is 24% per annum, find the monthly instalment.

16. A DVD player was purchsed by the customer with a cash down payment of ₹ 2750 and agreed to pay 3 equal half yearly instalments of ₹ 331 each. If the interest charged was 20% p.a. compounded half yearly, then find the cash price of the DVD player.

17. A flat can be purchased for ₹ 200000 cash from a housing society or on the terms that ₹ 67600 be paid in the beginning as cash down payment followed by three equal half yearly instalments. If the society charges interest at the rate of 20% per annum compounded semi-annually. If the flat is purchased under instalment plan, find each instalment.

18. A scooter was sold by a shopkeeper for cash down payment of ₹ 11000 alongwith 2 equal annual instalments of ₹ 6250 each. If the rate of interest charged was 25% per annum compounded annually, find the cash price of the scooter.

19. A computer is available for ₹ 78600 cash or for ₹ 25640 cash down payment and three equal quarterly instalments. If the dealer charges interest at the rate of 20% per annum compounded quarterly, find the value of each instalment.

20. A builder announces sale of flats each for ₹ 3000000 cash or ₹ 1031600 cash down payment and three equal quarterly instalments. If the rate of interest charged is 10% per annum compounded quarterly, compute the value of each instalment under the instalment scheme. Also find the total interest.

ANSWERS TO CHECK YOUR PROGRESS

9.1

1. 42.87%   2. $44\frac{4}{9}$ 3. $21\frac{1}{19}\%$ 4. $17\frac{1}{7}\%$ 5. 4.69%

6. 51.1%   7. 47.06%

9.2

1. ₹ 4000   2. $\frac{200}{9}$ 3. ₹ 775.77

4. ₹ 1934.55   4. ₹ 77.6 approx.

9.3

1. ₹ 1150   2. ₹ 10,000   3. ₹ 2500

4. ₹ 970   5. ₹ 3580

9.4

1. ₹ 463.05   2. ₹ 7290   3. ₹ 30,000, ₹ 1976.50

4. ₹ 25000   5. ₹ 7000

ANSWERS TO TERMINAL EXERCISE

1. $19\frac{1}{21}\%$ 2. $17\frac{1}{7}\%$ 3. $33\frac{1}{3}$ 4. $33\frac{1}{3}$

5. ₹ 920   6. ₹ 63.35   7. 5   8. ₹ 4000

9. 20.7%   10. 26.43%   11. ₹ 146.12   12. ₹ 3000

13. ₹ 2850.86   14. ₹ 366 (Approx)   15. ₹ 220   16. ₹ 6060

17. ₹ 53240   18. ₹ 20,000   19. ₹ 19448

20. ₹ 689210, ₹ 99230

Secondary Course Mathematics

Practice Work-Commercial Mathematics

Maximum Marks: 25 **Time : 45 Minutes**

Instructions:

1. Answer all the questions on a separate sheet of paper.
2. Give the following informations on your answer sheet

   Name

   Enrolment number

   Subject

   Topic of practice work

   Address
3. Get your practice work checked by the subject teacher at your study centre so that you get positive feedback about your performance.

Do not send practice work to National Institute of Open Schooling

1. By selling a school bag to a customer for ₹ 660, a shopkeeper makes a profit of 10%. The cost price (in rupees) of the school bag is 1

   (A) 625 (B) 600

   (C) 575 (D) 550

2. A customer purchases a radio set for ₹ 5400 after getting 10% discount on its list price. The list price of the radio set is 1

   (A) ₹ 5050 (B) ₹ 5800

   (C) ₹ 5950 (D) ₹ 6000

3. List price of a book is ₹ 300. A student purchases the book for ₹ 234. Percentage of discount is 1

   (A) 25 (B) 24

   (C) 22 (D) 20

4. The ratio (in simplest form) of 35 cm to 2 m is 1

 (A) 35: 2 (B) 35:200

 (C) 7:40 (D) 40:7

5. The difference in simple and compound interest for ₹ 2000 at 10% per annum in 2 years, compounded annually is 1

 (A) ₹ 20 (B) ₹ 200

 (C) ₹ 400 (D) ₹ 0

6. Determine the value of k if 20 : k :: 25 : 450. 2

7. If 120 is reduced to 96, what is the percentage reduction? 2

8. If the cost price of 15 articles is the same as the selling price of 12 articles, find the gain or loss percent in the transaction. 2

9. Find the single discount equivalent to the discount series of 20%, 15% and 10%. 2

10. Find the the sum of money which will amount to ₹ 26010 in six months at the rate of 8% per annum, when interest is compounded quarterly. 2

11. A sewing machine is available for ₹ 2600 cash or under instalment plan for ₹ 1000 cash down payment followed by 3 monthly instalments of ₹ 550 each. Find the rate of interest charged under the instalment plan. 4

12. A tree gains its height at the rate of 2% of what it was in the beginning of the month. If its height was 1.5 m in the beginning of January 2010, find the height at the end of April 2010.

10

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LINES AND ANGLES

Observe the top of your desk or table. Now move your hand on the top of your table. It gives an idea of a plane. Its edges give an idea of a line, its corner, that of a point and the edges meeting at a corner give an idea of an angle.

OBJECTIVES

After studying this lesson, you will be able to

- *illustrate the concepts of point, line, plane, parallel lines and interesecting lines;*
- *recognise pairs of angles made by a transversal with two or more lines;*
- *verify that when a ray stands on a line, the sum of two angles so formed is 180^0;*
- *verify that when two lines intersect, vertically opposite angles are equal;*
- *verify that if a transversal intersects two parallel lines then corresponding angles in each pair are equal;*
- *verify that if a transversal intersects two parallel lines then*

  (a) alternate angles in each pair are equal

  (b) interior angles on the same side of the transversal are supplementary;
- *prove that the sum of angles of a triangle is 180^0*
- *verify that the exterior angle of a triangle is equal to the sum of two interior opposite angles; and*
- *explain the concept of locus and exemplify it through daily life situations.*
- *find the locus of a point equidistent from (a) two given points, (b) two intersecting lines.*
- *solve problems based on starred result and direct numerical problems based on unstarred results given in the curriculum.*

EXPECTED BACKGROUND KNOWLEDGE

- point, line, plane, intersecting lines, rays and angles.
- parrallel lines

10.1 POINT, LINE AND ANGLE

In earlier classes, you have studied about a point, a line, a plane and an angle. Let us quickly recall these concepts.

Point : If we press the tip of a pen or pencil on a piece of paper, we get a fine dot, which is called a point.

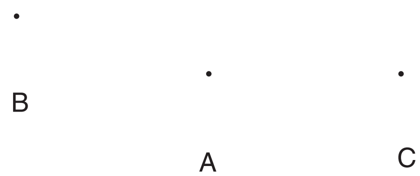


Fig. 10.1

A point is used to show the location and is represented by capital letters A, B, C etc.

10.1.1 Line

Now mark two points A and B on your note book. Join them with the help of a ruler or a scale and extend it on both sides. This gives us a straight line or simply a line.

Fig. 10.2

In geometry, a line is extended infinitely on both sides and is marked with arrows to give this idea. A line is named using any two points on it, viz, AB or by a single small letter *l*, *m* etc. (See fig. 10.3)

Fig. 10.3

The part of the line between two points A and B is called a line segment and will be named AB.

Observe that a line segment is the shortest path between two points A and B. (See Fig. 10.4)

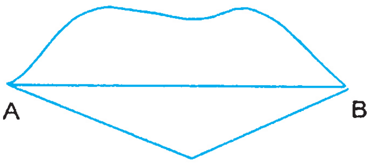


Fig. 10.4

10.1.2 Ray

If we mark a point X and draw a line, starting from it extending infinitely in one direction only, then we get a ray XY.

Fig. 10.5

X is called the initial point of the ray XY.

10.1.3 Plane

If we move our palm on the top of a table, we get an idea of a plane.

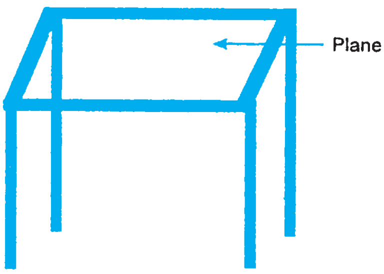


Fig. 10.6

Similarly, floor of a room also gives the idea of part of a plane.

Plane also extends infintely lengthwise and breadthwise.

Mark a point A on a sheet of paper.

How many lines can you draw passing though this point? As many as you wish.

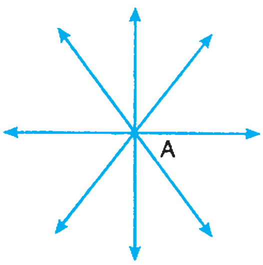


Fig. 10.7

In fact, we can draw an infinite number of lines through a point.

Take another point B, at some distance from A. We can again draw an infinite number of lines passing through B.

Fig. 10.8

Out of these lines, how many pass through both the points A and B? Out of all the lines passing through A, only one passes through B. Thus, only one line passes through both the points A and B. We conclude that **one and only one line can be drawn passing through two given points.**

Now we take three points in plane.

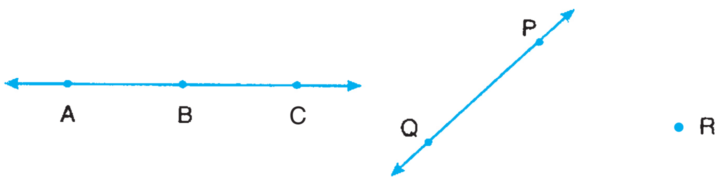


Fig. 10.9

We observe that a line may or may not pass through the three given points.

If a line can pass through three or more points, then these points are said to be **collinear.** For example the points A, B and C in the Fig. 10.9 are collinear points.

If a line **can not** be drawn passing through all three points (or more points), then they are said to be **non-collinear.** For example points P, Q and R, in the Fig. 10.9, are non-collinear points.

Since two points always lie on a line, we talk of collinear points only when their number is three or more.

Let us now take two distinct lines AB and CD in a plane.

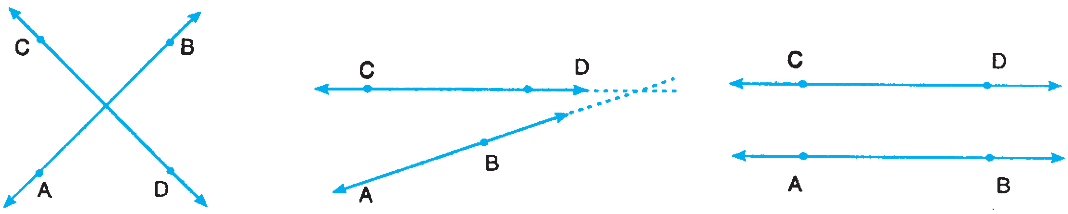
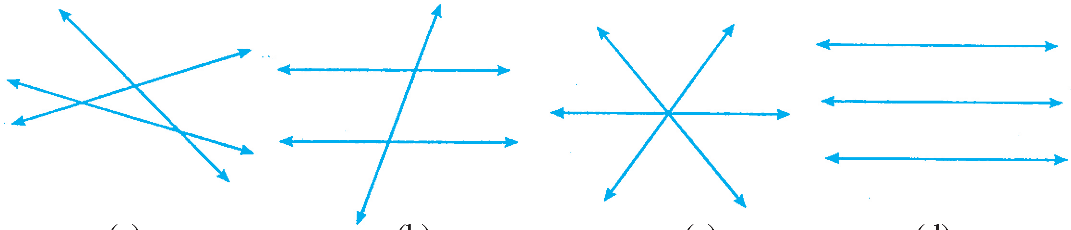


Fig. 10.10

How many points can they have in common? We observe that these lines can have. either (i) one point in common as in Fig. 10.10 (a) and (b). [In such a case they are called intersecting lines] or (ii) no points in common as in Fig. 10.10 (c). In such a case they are called **parrallel lines**.

Now observe three (or more) distinct lines in plane.



(a) (b) (c) (d)

Fig. 10.11

What are the possibilities ?

(i) They may interest in more than one point as in Fig. 10.11 (a) and 10.11 (b).

or (ii) They may intesect in one point only as in Fig. 10.11 (c). In such a case they are called concurrent lines.

or (iii) They may be non intersecting lines parrallel to each other as in Fig. 10.11 |(d).

10.1.4 Angle

Mark a point O and draw two rays OA and OB starting from O. The figure we get is called an angle. Thus, an angle is a figure consisting of two rays starting from a common point.

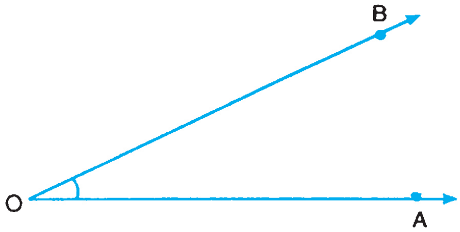


Fig. 10.11(A)

This angle may be named as angle AOB or angle BOA or simply angle O; and is written as $\angle AOB$ or $\angle BOA$ or $\angle O$. [see Fig. 10.11A]

An angle is measured in degrees. If we take any point O and draw two rays starting from it in opposite directions then the measure of this angle is taken to be 180^0 degrees, written as 180^0.

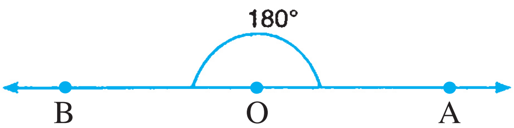


Fig. 10.12

This measure divided into 180 equal parts is called one degree (written as 1^o).

Angle obtained by two opposite rays is called a **straight angle**.

An angle of 90^0 is called a **right angle**, for example $\angle BOA$ or $\angle BOC$ is a right angle in Fig. 10.13.

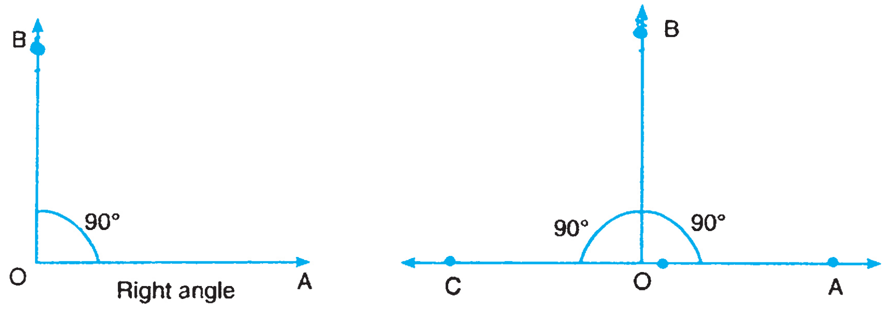
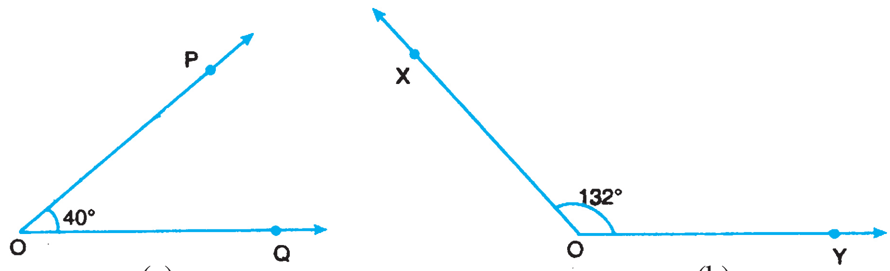


Fig. 10.13

Two lines or rays making a right angle with each other are called **perpendicular lines**. In Fig. 10.13 we can say OA is perpendicular to OB or vice-versa.

An angle less than 90^0 is called an **acute angle**. For example $\angle POQ$ is an acute angle in Fig. 10.14(a).

An angle greater than 90^0 but less than 180^0 is called an **obtuse angle**. For example, $\angle XOY$ is an obtuse angle in Fig. 10.14(b).



(a) (b)

Fig. 10.14

10.2 PAIRS OF ANGLES

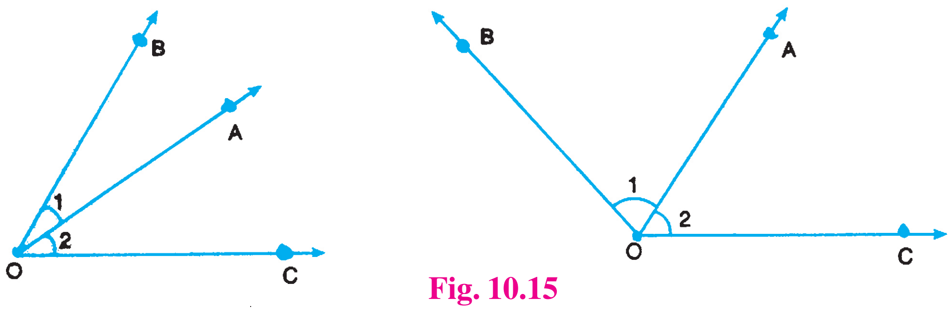


Fig. 10.15

Observe the two angles ∠1 and ∠2 in each of the figures in Fig. 10.15. Each pair has a common vertex O and a common side OA in between OB and OC. Such a pair of angles is called a '**pair of adjacent angles**'.

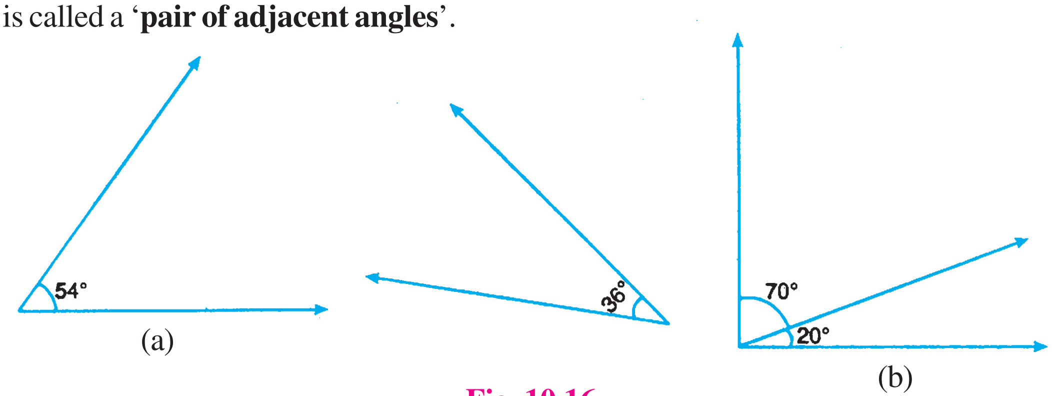


Fig. 10.16

Observe the angles in each pair in Fig. 10.16[(a) and (b)]. They add up to make a total of 90°.

A pair of angles, whose sum is 90°, is called a pair of **complementary angles**. Each angle is called the **complement** of the other.

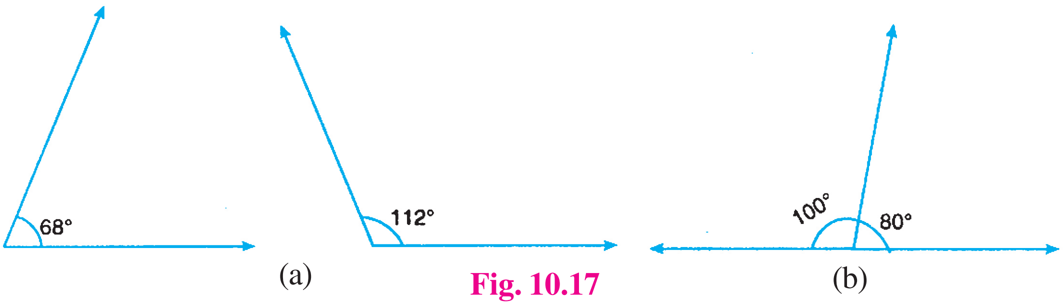


Fig. 10.17

Again observe the angles in each pair in Fig. 10.17[(a) and (b)].

These add up to make a total of 180°.

A pair of angles whose sum is 180°, is called a pair of supplementary angles.

Each such angle is called the **supplement** of the other.

Draw a line AB. From a point C on it draw a ray CD making two angles ∠X and ∠Y.

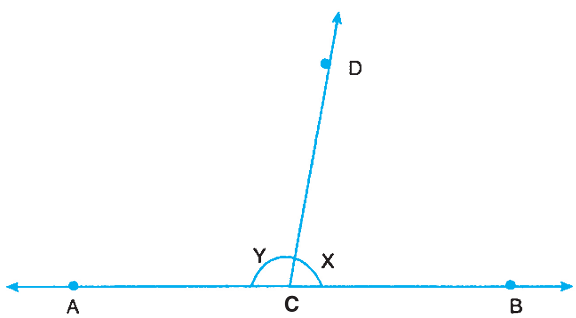


Fig. 10.18

If we measure $\angle X$ and $\angle Y$ and add, we will always find the sum to be 180^0, whatever be the position of the ray CD. We conclude

If a ray stands on a line then the sum of the two adjacent angles so formed is 180^o.

The pair of angles so formed as in Fig. 10.18 is called a **linear pair** of angles.

Note that they also make a pair of supplementary angles.

Draw two intersecting lines AB and CD, intersecting each other at O.

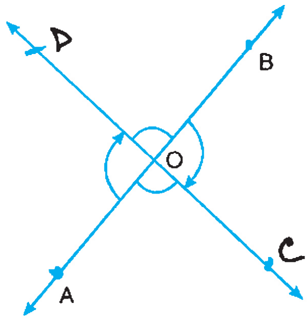


Fig. 10.19

$\angle AOC$ and $\angle DOB$ are angles opposite to each other. These make a pair of **vertically oppposite angles**. Measure them. You will always find that

$\angle AOC = \angle DOB$.

$\angle AOD$ and $\angle BOC$ is another pair of vertically opposite angles. On measuring, you will again find that

$\angle AOD = \angle BOC$

We conclude :

If two lines intersect each other, the pair of vertically opposite angles are equal.

An activity for you.

Attach two strips with a nail or a pin as shown in the figure.

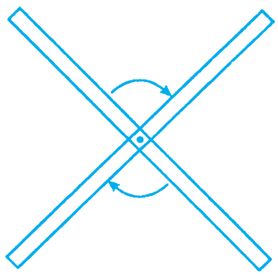


Fig. 10.20

Rotate one of the strips, keeping the other in position and observe that the pairs of vertically opposite angles thus formed are always equal.

A line which intersects two or more lines at distinct points is called a **transversal**. For example line *l* in Fig. 10.21 is a transversal.

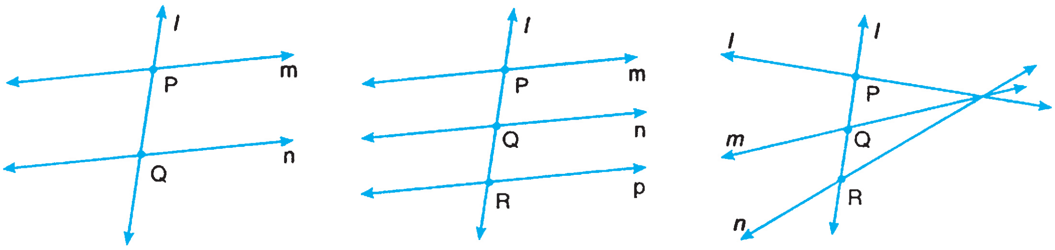


Fig. 10.21

When a transversal intersects two lines, eight angles are formed.

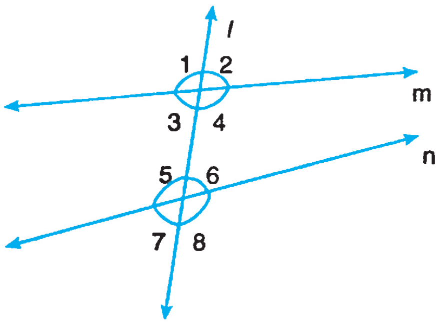


Fig. 10.22

These angles in pairs are very important in the study of properties of parallel lines. Some of the useful pairs are as follows :

(a) $\angle 1$ and $\angle 5$ is a pair of corresponding angles. $\angle 2$ and $\angle 6$, $\angle 3$ and $\angle 7$ and $\angle 4$ and $\angle 8$ are other pairs of corresponding angles.

(b) $\angle 3$ and $\angle 6$ is a pair of alternate angles. $\angle 4$ and $\angle 5$ is another pair of alternate angles.

(c) $\angle 3$ and $\angle 5$ is a pair of interior angles on the same side of the transversal.

$\angle 4$ and $\angle 6$ is another pair of interior angles.

In Fig. 10.22 above, lines m and n are not parallel; as such, there may not exist any relation between the angles of any of the above pairs. However, when lines are parallel, there are some very useful relations in these pairs, which we study in the following:

When a transversal intersects two parallel lines, eight angles are formed, whatever be the position of parallel lines or the transversal.

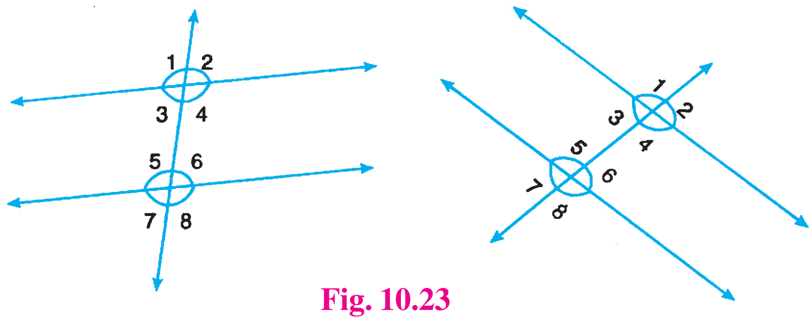


Fig. 10.23

If we measure the angles, we shall alwys find that

$$\angle 1 = \angle 5, \qquad \angle 2 = \angle 6, \qquad \angle 3 = \angle 7 \text{ and } \angle 4 = \angle 8$$

that is, angles in each pair of corresponding angles are equal.

Also $\qquad \angle 3 = \angle 6$ and $\angle 4 = \angle 5$

that is, angles in each pair of alternate angle are equal.

Also, $\qquad \angle 3 + \angle 5 = 180^\circ$ and $\angle 4 + \angle 6 = 180^\circ$.

Hence we conclude :

When a transversal intersects two parallel lines, then angles in

(i) each pair of corresponding angles are equal

(ii) each pair of alternate angles are equal

(iii) each pair of interior angles on the same side of the transversal are supplementary,

You may also verify the truth of these results by drawing a pair of parallel lines (using parallel edges of your scale) and a transversal and measuring angles in each of these pairs.

Converse of each of these results is also true. To verify the truth of the first converse, we draw a line AB and mark two points C and D on it.

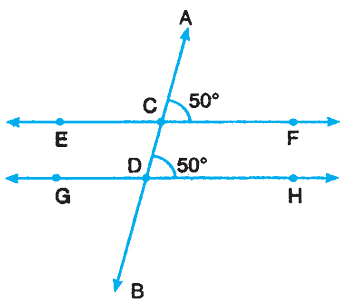


Fig. 10.24

At C and D, we construct two angles ACF and CDH equal to each other, say 50°, as shown in Fig. 10.24. On producing EF and GH on either side, we shall find that they do not intersect each other, that is, they are parallel.

In a similar way, we can verify the truth of the other two converses.

Hence we conclude that

When a transversal inersects two lines in such a way that angles in

(i) any pair of corresponding angles are equal

or (ii) any pair of alternate angles are equal

or (iii) any pair of interior angles on the same side of transversal are supplementary then the two lines are parallel.

Example 10.1 : Choose the correct answwer out of the alternative options in the following multiple choice questions.

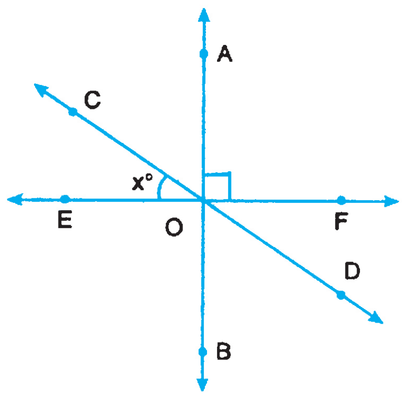


Fig. 10.25

(i) In Fig. 10.25, $\angle FOD$ and $\angle BOD$ are

(A) supplementary angles (B) complementary angles

(C) vertically opposite angles (D) a linear pair of angles **Ans.** (B)

(ii) In Fig. 10.25, $\angle COE$ and $\angle BOE$ are

(A) complementary angles (B) supplementary angles

(C) a linear pair (D) adjacent angles **Ans.** (D)

(iii) In Fig. 10.25, $\angle BOD$ is equal to

(A) x^o (B) $(90 + x)^o$

(C) $(90 - x)^o$ (D) $(180 - x)^o$ **Ans** (C)

(iv) An angle is 4 times its supplement; the angle is

(A) 39^o (B) 72^o

(C) 108^o (D) 144^o **Ans** (D)

(v) What value of x will make ACB a straight angle in Fig. 10.26

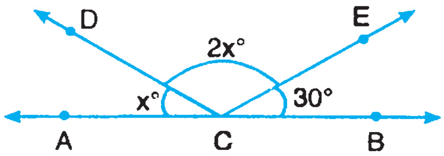


Fig. 10.26

(A) 30° (B) 40°

(C) 50° (D) 60° **Ans** (C)

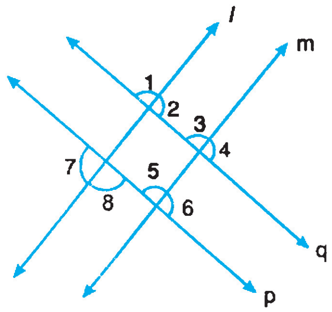


Fig. 10.27

In the above figure, *l* is parallel to m and p is parallel to q.

(vi) $\angle 3$ and $\angle 5$ form a pair of

(A) Alternate angles (B) interior angles

(C) vertically opposite (D) corresponding angles **Ans** (D)

(vii) In Fig. 10.27, if $\angle 1 = 80°$, then $\angle 6$ is equal to

(A) 80° (B) 90°

(C) 100° (D) 110° **Ans** (C)

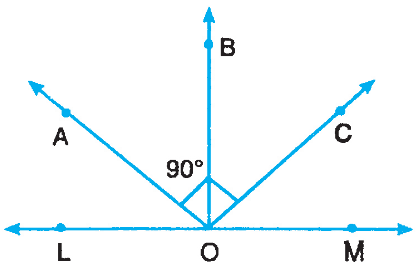


Fig. 10.28

(viii) In Fig. 10.28, OA bisects $\angle LOB$, OC bisects $\angle MOB$ and $\angle AOC = 90^0$. Show that the points L, O and M are collinear.

Solution : $\angle BOL = 2\ \angle BOA$ ...(i)

and $\angle BOM = 2\ \angle BOC$ ...(ii)

Adding (i) and (ii), $\angle BOL + \angle BOM = 2\ \angle BOA + 2\angle BOC$

$\therefore\ \angle LOM = 2[\angle BOA + \angle BOC]$

$= 2 \times 90^{o}$

$= 180^{o}$ = a straight angle

$\therefore$ L, O and M are collinear.

CHECK YOUR PROGRESS 10.1.

1. Choose the correct answer out of the given alternatives in the following multiple choice questions :

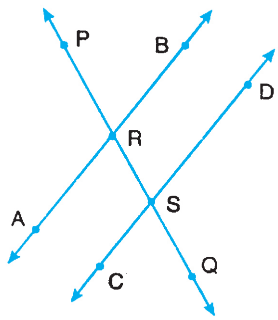


Fig. 10.29

In Fig. 10.29, AB || CD and PQ intersects them at R and S respectively.

(i) $\angle ARS$ and $\angle BRS$ form

(A) a pair of alternate angles

(B) a linear pair

(C) a pair of corresponding angles

(D) a pair of vertically opposite angles

(ii) $\angle ARS$ and $\angle RSD$ form a pair of

(A) Alternate angles (B) Vertically opposite angles

(C) Corresponding angles (D) Interior angles

(iii) If $\angle PRB = 60^{o}$, then $\angle QSC$ is

(A) 120^{o} (B) 60^{o}

(C) 30^o (D) 90^o

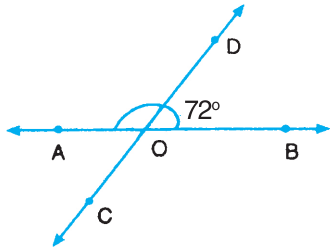


Fig. 10.30

(iv) In Fig. 10.30 above, AB and CD intersect at O. ∠COB is equal to

(A) 36^o (B) 72^o

(C) 108^o (D) 144^o

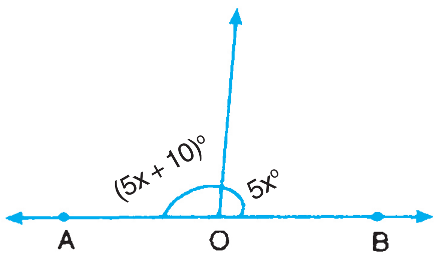


Fig. 10.31

2. In Fig. 10.31 above, AB is a straight line. Find x

3. In Fig. 10.32 below, *l* is parallel to m. Find angles 1 to 7.

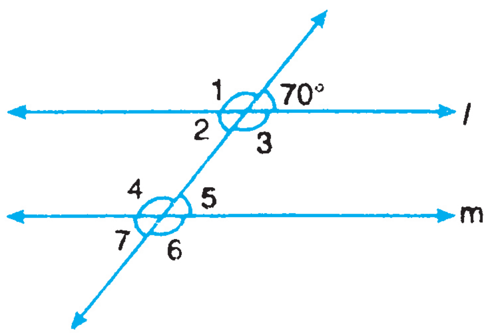


Fig. 10.32

10.3 TRIANGLE, ITS TYPES AND PROPERTIES

Triangle is the simplest polygon of all the closed figures formed in a plane by three line segments.

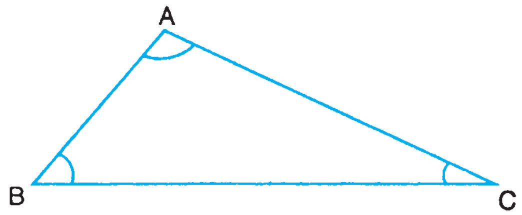


Fig. 10.33

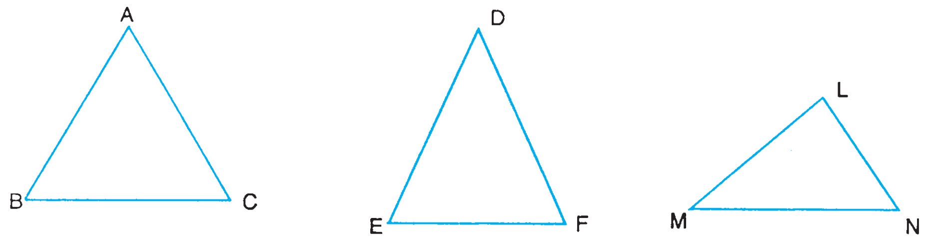
It is a closed figure formed by three line segments having six elements, namely three **angles** (i) ∠ABC or ∠B (ii) ∠ACB or ∠C (iii) ∠CAB or ∠A and three **sides** : (iv) AB (v) BC (vi) CA

It is named as Δ ABC or Δ BAC or Δ CBA and read as triangle ABC or triangle BAC or triangle CBA.

10.3.1 Types of Triangles

Triangles can be classified into different types in two ways.

(a) On the basis of sides



(i) (ii) (iii)

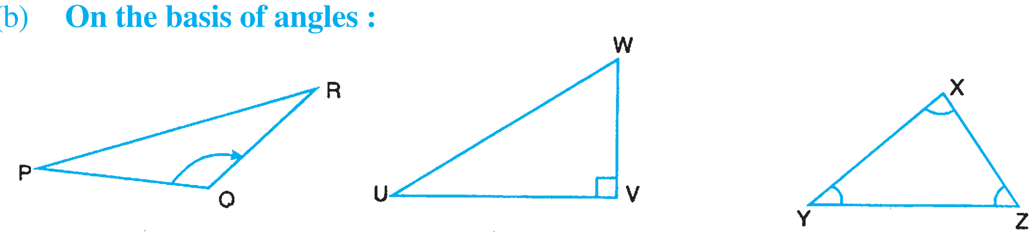
Fig. 10.34

(i) **Equilateral triangle :** a triangle in which all the three sides are equal is called an equilateral trangle. [Δ ABC in Fig. 10.34(i)]

(ii) **Isosceles triangle :** A triangle in which two sides are equal is called an isosceles triangle. [ΔDEF in Fig. 10.34(ii)]

(iii) **Scalene triangle :** A triangle in which all sides are of different lengths, is called a sclene triangle [Δ LMN in Fig. 10.34(iii)]

(b) **On the basis of angles :**



(i) (ii) (iii)

Fig. 10.35

(i) **Obtuse angled triangle :** A triangle in which one of the angles is an obtuse angle is called an **obtuse angled triangle** or simply obtuse triangle [Δ PQR is Fig. 10.35(i)]

(ii) **Right angled triangle :** A triangle in which one of the angles is a right angle is called a **right angled triangle** or right triangle. [Δ UVW in Fig. 10.35(ii)]

(iii) **Acute angled triangle :** A triangle in which all the three angles are acute is called an **acute angled triangle** or acute triangle [Δ XYZ in Fig. 10.35(iii)

Now we shall study some important properties of angles of a triangle.

10.3.2 Angle Sum Property of a Triangle

We draw two triangles and measure their angles.

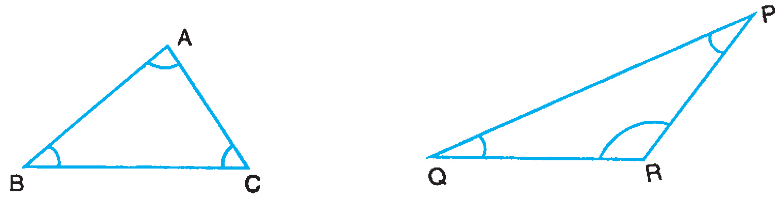


Fig. 10.36

In Fig. 10.36 (a), $\angle A = 80^o$, $\angle B = 40^o$ and $\angle C = 60^o$

$\therefore \quad \angle A + \angle B + \angle C = 80^o + 40^o + 60^o = 180^o$

In Fig. 10.36(b), $\angle P = 30^o$, $\angle Q = 40^o$, $\angle R = 110^o$

$\therefore \quad \angle P + \angle Q + \angle R = 30^o + 40^o + 110^o = 180^o$

What do you observe? Sum of the angles of triangle in each case in 180^0.

We will prove this result in a logical way naming it as a theorem.

Theorem : The sum of the three angles of triangle is 180º.

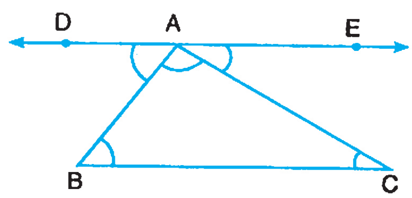


Fig. 10.37

Given : A triangle ABC

To Prove : $\angle A + \angle B + \angle C = 180^0$

Construction : Through A, draw a line DE parallel to BC.

Proof : Since DE is parallel to BC and AB is a transversal.

$\therefore \quad \angle B = \angle DAB$ (Pair of alternate angles)

Similarly $\angle C = \angle EAC$ (Pair of alternate angles)

$\therefore \angle B + \angle C = \angle DAB + \angle EAC$ ...(1)

Now adding $\angle A$ to both sides of (1)

$$\angle A + \angle B + \angle C = \angle A + \angle DAB + \angle EAC$$

$$= 180^\circ \quad \text{(Angles making a straight angle)}$$

10.3.3 Exterior Angles of a Triangle

Let us produce the side BC of ΔABC to a point D.

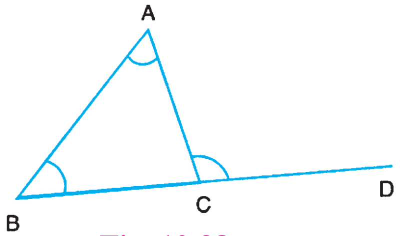


Fig. 10.38

In Fig. 10.39, observe that there are six exterior angles of the ΔABC, namely $\angle 1$, $\angle 2$, $\angle 3$, $\angle 4$, $\angle 5$ and $\angle 6$.

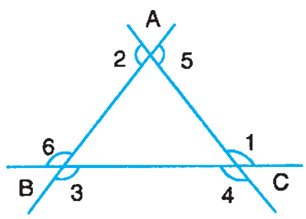


Fig. 10.39

In Fig. 10.38, $\angle ACD$ so obtained is called an exterior angle of the ΔABC. Thus,

The angle formed by a side of the triangle produced and another side of the triangle is called an exterior angle of the triangle.

Corresponding to an exterior angle of a triangle, there are two interior opposite angles.

Interior opposite angles are the angles of the triangle not forming a linear pair with the given exterior angle.

For example in Fig. 10.38, $\angle A$ and $\angle B$ are the two interior opposite angles corresponding to the exterior angle ACD of ΔABC. We measure these angles.

$$\angle A = 60^\circ$$

$$\angle B = 50^\circ$$

and $\angle ACD = 110^o$

We observe that $\angle ACD = \angle A + \angle B$.

This observation is true in general.

Thus, we may conclude :

An exterior angle of a triangle is equal to the sum of the two interior opposite angles.

Examples 10.3 : Choose the correct answer out of the given alternatives in the following multiple choice questions:

(i) Which of the following can be the angles of a triangle?

(A) 65^o, 45^o and 80^o (B) 90^o, 30^o and 61^o

(C) 60^o, 60^o and 59^o (D) 60^o, 60^o and 60^o. **Ans** (D)

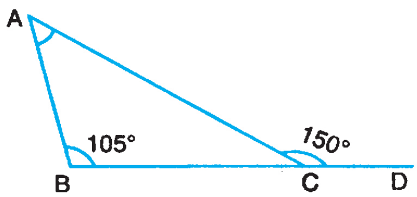


Fig. 10.40

(ii) In Fig. 10.40 $\angle A$ is equal to

(A) 30^o (B) 35^o

(C) 45^o (D) 75^o **Ans** (C)

(iii) In a triangle, one angle is twice the other and the third angle is 60^0. Then the largest angle is

(A) 60^o (B) 80^o

(C) 100^o (D) 120^o **Ans** (B)

Example 10.4:

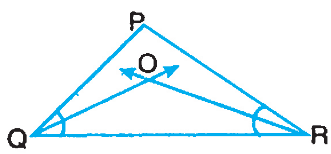


Fig. 10.41

In Fig. 10.41, bisctors of $\angle PQR$ and $\angle PRQ$ intersect each other at O. Prove that

$\angle QOR = 90^o + \frac{1}{2}\angle P$.

Solution : $\angle QOR = 180^o - \frac{1}{2}[\angle PQR + \angle PRQ)]$

$$= 180^o - \frac{1}{2}(\angle PQR + \angle PRQ)$$

$$= 180^o - \frac{1}{2}(180^o \quad - \angle P)$$

$$= 180^o - 90^o + \frac{1}{2}\angle P = 90^o + \frac{1}{2}\angle P$$

CHECK YOUR PROGRESS 10.2

1. Choose the correct answer out of given alternatives in the following multiple choice questions:

(i) A triangle can have

(A) Two right angles (B) Two obtuse angles

(C) At the most two acute angles (D) All three acute angles

(ii) In a right triangle, one exterior angles is 120^0, The smallest angle of the triangles is

(A) 20^o (B) 30^0

(C) 40^o (D) 60^0

(iii)

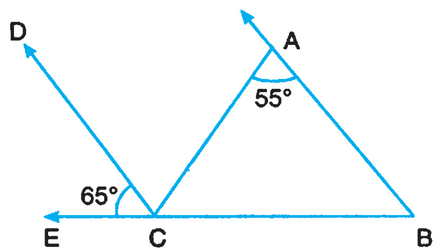


Fig. 10.42

In Fig. 10.42, CD is parallel to BA. $\angle ACB$ is equal to

(A) 55^o (B) 60^o

(C) 65^o (D) 70^o

2. The angles of a triangle are in the ratio 2 : 3 : 5, find the three angles.

3. Prove that the sum of the four angles of a quadrilateral is 360^o.

4. In Fig. 10.43, ABCD is a trapezium such that AB||DC. Find ∠D and ∠C and verify that sum of the four angles is 360^o.

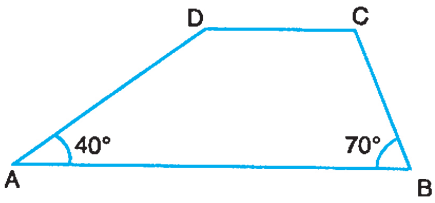


Fig. 10.43

5. Prove that if one angle of a triangle is equal to the sum of the other two angles, then it is a right triangle.

6. In Fig. 10.44, ABC is triangle such that ∠ABC = ∠ACB. Find the angles of the triangle.

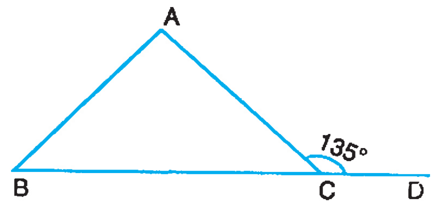


Fig. 10.44

10.4 LOCUS

During the game of cricket, when a player hits the ball, it describes a path, before being caught or touching the ground.

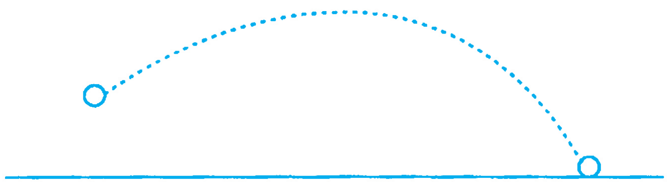


Fig. 10.44

The path described is called Locus.

A figure in geometry is a result of the path traced by a point (or a very small particle) moving under certain conditions.

For example:

(1) Given two parallel lines l and m, also a point P between them equidistant from both the lines.

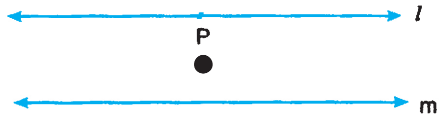


Fig. 10.45

If the particle moves so that it is equidistant from both the lines, what will be its path?

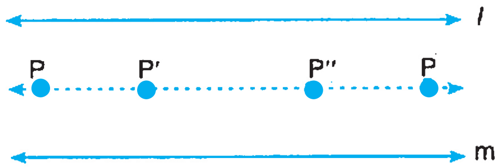


Fig. 10.46

The path traced by P will be a line parallel to both the lines and exactly in the middle of them as in Fig. 10.46.

(2) Given a fixed point O and a point P at a fixed distance *d*.

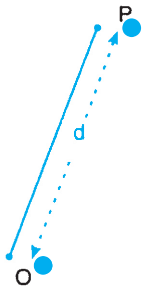


Fig. 10.47

If the point P moves in a plane so that it is always at a constant distance *d* from the fixed point O, what will be its path?

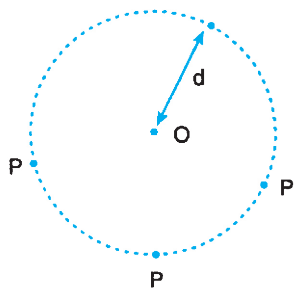


Fig. 10.48

The path of the moving point P will be a circle as shown in Fig. 10.48.

(3) Place a small piece of chalk stick or a pebble on top of a table. Strike it hard with a pencil or a stick so that it leaves the table with a certain speed and observe its path after it leaves the table.

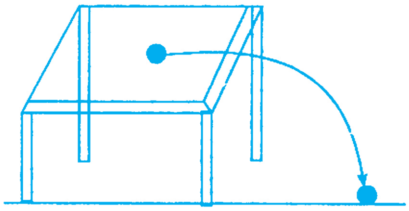


Fig. 10.49

The path traced by the pebble will be a curve (part of what is known as a parabola) as shown in Fig. 10.49.

Thus, locus of a point moving under certain conditions is the path or the geometrical figure, every point of which satisfies the given conditon(s).

10.4.1 Locus of a point equidistant from two given points

Let A and B be the two given points.

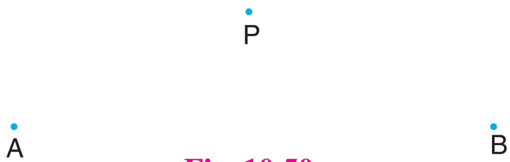


Fig. 10.50

We have to find the locus of a point P such that PA = PB.

Joint AB. Mark the mind point of AB as M. Clearly, M is a point which is equidistant from A and B. Mark another point P using compasses such that PA = PB. Join PM and extend it on both sides. Using a pair of divider or a scale, it can easily be verified that every point on PM is equidistant from the points A and B. Also, if we take any other point Q not lying on line PM, then $QA \neq QB$.

Also $\angle AMP = \angle BMP = 90^{\circ}$

That is, PM is the perpendicular bisector of AB.

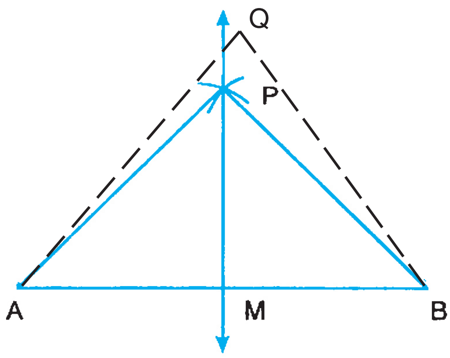


Fig. 10.51

Thus, we may conclude the following:

The locus of a point equidistant from two given poitns is the perpendicular bisector of the line segment joining the two points.

Activity for you :

Mark two points A and B on a sheet of paper and join them. Fold the paper along mid-point of AB so that A coincides with B. Make a crease along the line of fold. This crease is a straight line. This is the locus of the point equidistant from the given points A and B. It can be easily checked that very point on it is equidistant from A and B.

10.4.2 Locus of a point equidistant from two lines intersecting at O

Let AB and CD be two given lines intersecting at O.

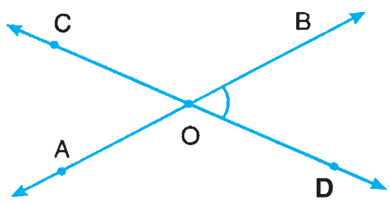


Fig. 10.52

We have to find the locus of a point P which is equidistant from both AB and CD.

Draw bisectors of $\angle BOD$ and $\angle BOC$.

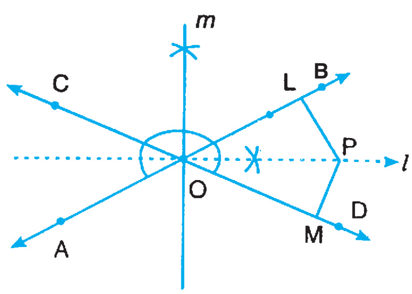


Fig. 10.53

If we take any point P on any bisector l or m, we will find perpendicular distances PL and PM of P from the lines AB and CD are equal.

that is, $PL = PM$

If we take any other point, say Q, not lying on any bisector l or m, then QL will not be equal to QM.

Thus, we may conclude :

The locus of a point equidistant from two intersecting lines is the pair of lines, bisecting the angles formed by the given lines.

Activity for you :

Draw two lines AB and CD intersecting at O, on a sheet of paper. Fold the paper through O so that AO falls on CO and OD falls on OB and mark the crease along the fold. Take a piont P on this crease which is the bisector of ∠BOD and check using a set square that

PL = PM

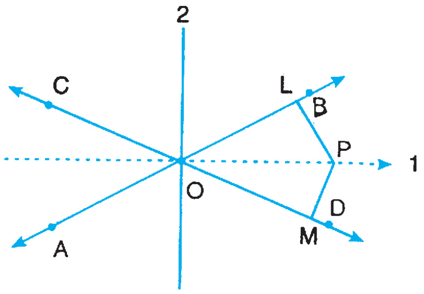


Fig. 10.54

In a similar way find the other bisector by folding again and getting crease 2. Any point on this crease 2 is also equidistant from both the lines.

Example 10.5 : Find the locus of the centre of a circle passing through two given points.

Solution : Let the two given points be A and B. We have to find the position or positions of centre O of a circle passing through A and B.

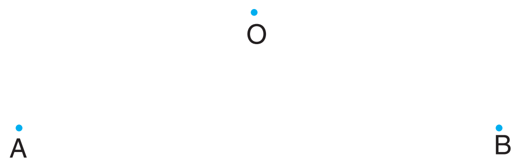


Fig. 10.55

Point O must be equidistant from both the points A and B. As we have already learnt, the locus of the point O will be the perpendicular bisector of AB.

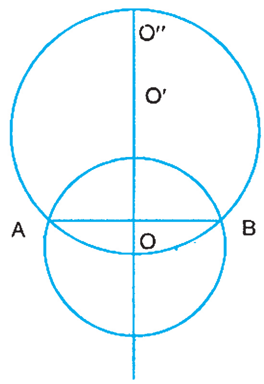


Fig. 10.56

CHECK YOU PROGRESS 10.3

1. Find the locus of the centre of a circle passing through three given points A, B and C which are non-collinear.
2. There are two villages certain distance apart. A well is to be dug so that it is equidistant from the two villages such that its distance from each village is not more than the distance between the two villages. Representing the villages by points A and B and the well by point P. show in a diagram the locus of the point P.
3. Two straight roads AB and CD are intersecting at a point O. An observation post is to be constructred at a distance of 1 km from O and equidistant from the roads AB and CD. Show in a diagram the possible locations of the post.
4. Find the locus of a point which is always at a distance 5 cm from a given line AB.

LET US SUM UP

- A line extends to inifinity on both sides and a line segment is only a part of it between two points.
- Two distinct lines in a plane may either be intersecting or parallel.
- If three or more lines intersect in one point only then they are called cocurrent lines.
- Two rays starting from a common point form an angle.
- A pair of angles, whose sum is 90^0 is called a pair of complementary angles.
- A pair of angles whose sum is 180^0 is called a pair of supplementary angles.
- If a ray stands on a line then the sum of the two adjacent angles, so formed is 180^0
- If two lines intersect each other the pairs of vertically opposite angles are equal
- When a transversal intersects two parallel lines, then

  (i) corresponding angles in a pair are equal.

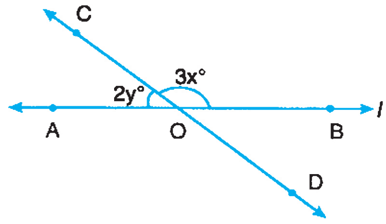
  (ii) alternate angles are equal.

  (iii) interior angles on the same side of the transversal are supplementary.
- The sum of the angles of a triangle is 180^0
- An exterior angle of a triangle to equal to the sum of the two interior opposite angles
- Locus of a point equidistant from two given points is the perpendicular bisector of the line segment joing the points.

- The locus of a point equidistant from the intersecting lines is the pair of lines, bisecting the angle formed by the given lines.

TERMINAL EXERCISE

1. In Fig. 10.57, if x = 42, then determine (a) y (b) $\angle AOD$



2.

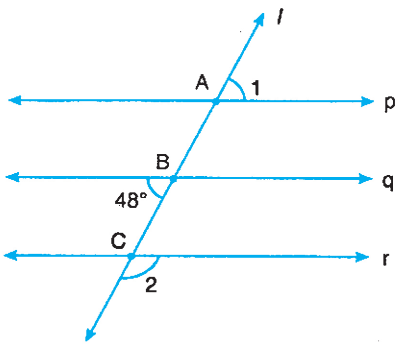


Fig. 10.58

In the above figure p, q and r are parallel lines intersected by a transversal l at A, B and C respectively. Find $\angle 1$ and $\angle 2$.

3. The sum of two angles of a triangle is equal to its third angle. Find the third angle. What type of triangle is it?

4.

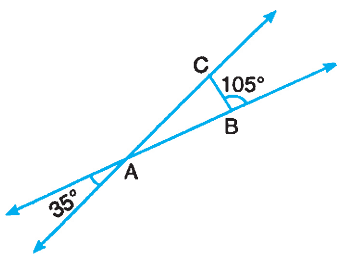


Fig. 10.59

In Fig. 10.59, sides of Δ ABC have been produced as shown. Find the angles of the triangle.

5.

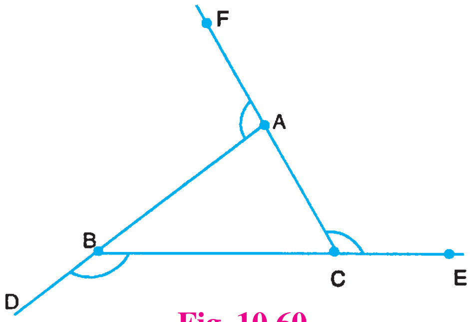


Fig. 10.60

In Fig. 10.60, sides AB, BC and CA of the triangle ABC have been produced as shown. Show that the sum of the exterior angles so formed is 360°.

6.

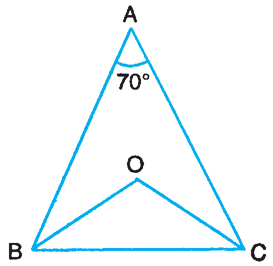


Fig. 10.61

In Fig. 10.61 ABC is a triangle in which bisectors of $\angle B$ and $\angle C$ meet at O. Show that $\angle BOC = 125°$.

7.

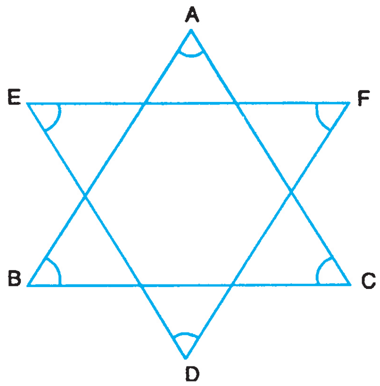


Fig. 10.62

In Fig. 10.62 above, find the sum of the angles, $\angle A$, $\angle F$, $\angle C$, $\angle D$, $\angle B$ and $\angle E$.

8.

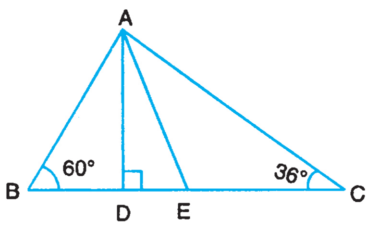


Fig. 10.63

In Fig. 10.63 in Δ ABC, AD is perpendicular to BC and AE is bisector of $\angle BAC$. Find $\angle DAE$,

9.

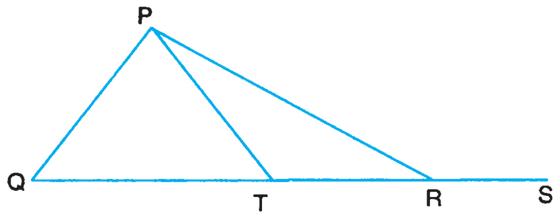


Fig. 10.64

In Fig. 10.64 above, in Δ PQR, PT is bisector of $\angle P$ and QR is produced to S. Show that $\angle PQR + \angle PRS = 2\angle PTR$.

10. Prove that the sum of the (interior) angles of a pentagon is 540^0.

11. Find the locus of a point equidistant from two parallel lines l and m at a distance of 5 cm from each other.

12. Find the locus of a point equidistant from points A and B and also equidistant from rays AB and AC of Fig. 10.65.

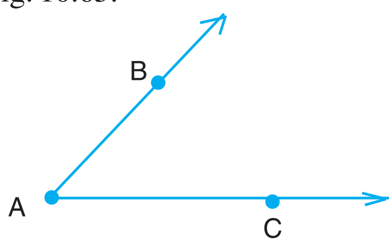


Fig. 10.65

ANSWERS TO CHECK YOUR PROGRESS

10.1

1. (i) (B) (ii) (A) (iii) (B) (iv) (C)

2. $x = 17^0$.

3. $\angle 1 = \angle 3 = \angle 4 = \angle 6 = 110^o$

and $\angle 2 = \angle 5 = \angle 7 = 70^o$.

10.2

1. (i) (D) (ii) (B) (iii) (B)

2. 36^o, 54^o and 90^o 4. $\angle D = 140^o$ and $\angle C = 110^o$

6. $\angle ABC = 45^o$, $\angle ACB = 45^o$ and $\angle A = 90^o$

10.3

1. Only a point, which is the point of intersection of perpendicular bisectors of AB and BC.
2. Let the villages be A and B, then locus will be the line segment PQ, perpendicular bisector of AB such that

   AP = BP = QA = QB = AB

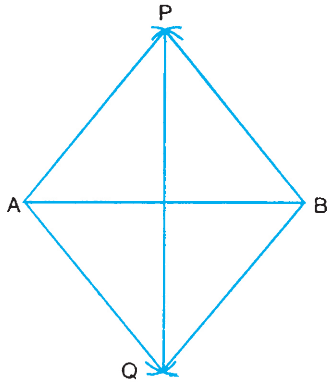


Fig. 10.65

3. Possible locations will be four points two points P and Q on the bisector of ∠AOC and two points R and S on the bisector of ∠BOC.

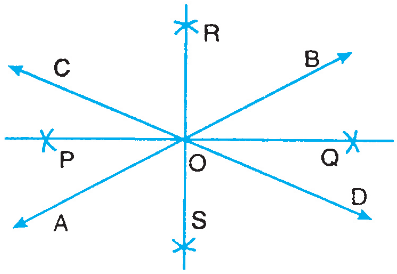


Fig. 10.66

4. Two on either side of AB and lines parallel to AB at a distance of 5 cm from AB.

ANSWERS TO TERMINAL EXERCISE

1. (a) y = 27 (b) = 126° 2. ∠1 = 48° and ∠2 = 132°

3. Third angle = 90°, Right triangle 4. ∠A = 35°, ∠B = 75° ∠C = 70°

7. 360° 8. 12°

11. A line parallel to locus *l* and m at a distance of 2.5 cm from each.

12. Point of intersection of the perpendicular bisector of AB and bisector of ∠BAC.

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10

LINES AND ANGLES

Observe the top of your desk or table. Now move your hand on the top of your table. It gives an idea of a plane. Its edges give an idea of a line, its corner, that of a point and the edges meeting at a corner give an idea of an angle.

OBJECTIVES

After studying this lesson, you will be able to

- *illustrate the concepts of point, line, plane, parallel lines and interesecting lines;*
- *recognise pairs of angles made by a transversal with two or more lines;*
- *verify that when a ray stands on a line, the sum of two angles so formed is 180^0;*
- *verify that when two lines intersect, vertically opposite angles are equal;*
- *verify that if a transversal intersects two parallel lines then corresponding angles in each pair are equal;*
- *verify that if a transversal intersects two parallel lines then*

  (a) alternate angles in each pair are equal

  (b) interior angles on the same side of the transversal are supplementary;
- *prove that the sum of angles of a triangle is 180^0*
- *verify that the exterior angle of a triangle is equal to the sum of two interior opposite angles; and*
- *explain the concept of locus and exemplify it through daily life situations.*
- *find the locus of a point equidistent from (a) two given points, (b) two intersecting lines.*
- *solve problems based on starred result and direct numerical problems based on unstarred results given in the curriculum.*

EXPECTED BACKGROUND KNOWLEDGE

- point, line, plane, intersecting lines, rays and angles.
- parrallel lines

10.1 POINT, LINE AND ANGLE

In earlier classes, you have studied about a point, a line, a plane and an angle. Let us quickly recall these concepts.

Point : If we press the tip of a pen or pencil on a piece of paper, we get a fine dot, which is called a point.

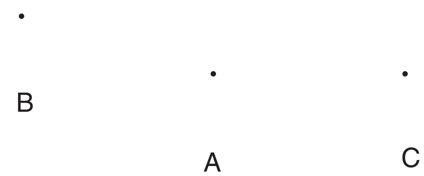


Fig. 10.1

A point is used to show the location and is represented by capital letters A, B, C etc.

10.1.1 Line

Now mark two points A and B on your note book. Join them with the help of a ruler or a scale and extend it on both sides. This gives us a straight line or simply a line.

Fig. 10.2

In geometry, a line is extended infinitely on both sides and is marked with arrows to give this idea. A line is named using any two points on it, viz, AB or by a single small letter *l*, *m* etc. (See fig. 10.3)

Fig. 10.3

The part of the line between two points A and B is called a line segment and will be named AB.

Observe that a line segment is the shortest path between two points A and B. (See Fig. 10.4)

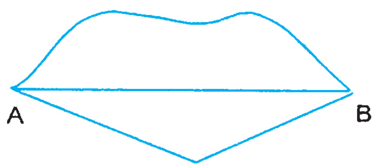


Fig. 10.4

10.1.2 Ray

If we mark a point X and draw a line, starting from it extending infinitely in one direction only, then we get a ray XY.

Fig. 10.5

X is called the initial point of the ray XY.

10.1.3 Plane

If we move our palm on the top of a table, we get an idea of a plane.

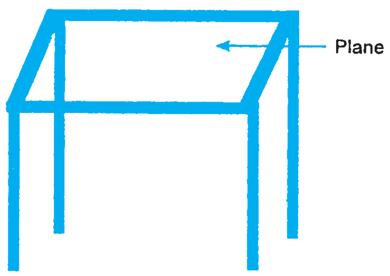


Fig. 10.6

Similarly, floor of a room also gives the idea of part of a plane.

Plane also extends infintely lengthwise and breadthwise.

Mark a point A on a sheet of paper.

How many lines can you draw passing though this point? As many as you wish.

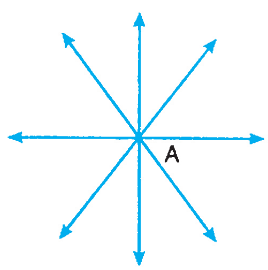


Fig. 10.7

In fact, we can draw an infinite number of lines through a point.

Take another point B, at some distance from A. We can again draw an infinite number of lines passing through B.

Fig. 10.8

Out of these lines, how many pass through both the points A and B? Out of all the lines passing through A, only one passes through B. Thus, only one line passes through both the points A and B. We conclude that **one and only one line can be drawn passing through two given points.**

Now we take three points in plane.

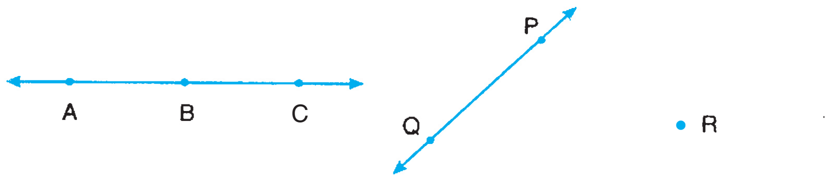


Fig. 10.9

We observe that a line may or may not pass through the three given points.

If a line can pass through three or more points, then these points are said to be **collinear.** For example the points A, B and C in the Fig. 10.9 are collinear points.

If a line **can not** be drawn passing through all three points (or more points), then they are said to be **non-collinear.** For example points P, Q and R, in the Fig. 10.9, are non-collinear points.

Since two points always lie on a line, we talk of collinear points only when their number is three or more.

Let us now take two distinct lines AB and CD in a plane.

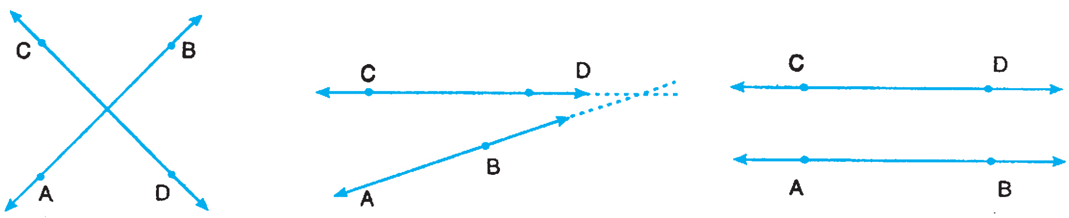


Fig. 10.10

How many points can they have in common? We observe that these lines can have. either (i) one point in common as in Fig. 10.10 (a) and (b). [In such a case they are called intersecting lines] or (ii) no points in common as in Fig. 10.10 (c). In such a case they are called **parrallel lines**.

Now observe three (or more) distinct lines in plane.

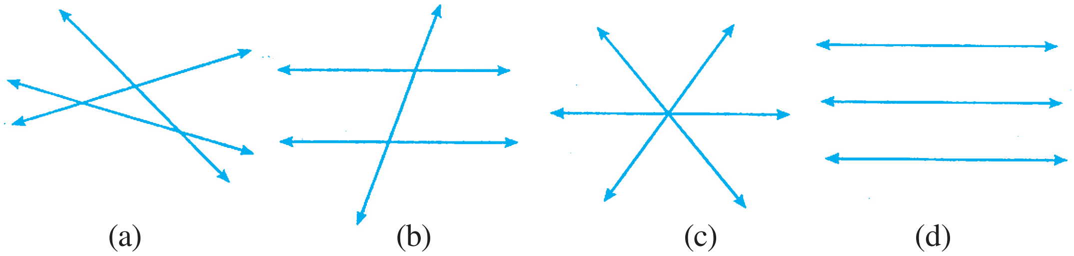


Fig. 10.11

What are the possibilities ?

(i) They may interest in more than one point as in Fig. 10.11 (a) and 10.11 (b).

or (ii) They may intesect in one point only as in Fig. 10.11 (c). In such a case they are called concurrent lines.

or (iii) They may be non intersecting lines parrallel to each other as in Fig. 10.11 |(d).

10.1.4 Angle

Mark a point O and draw two rays OA and OB starting from O. The figure we get is called an angle. Thus, an angle is a figure consisting of two rays starting from a common point.

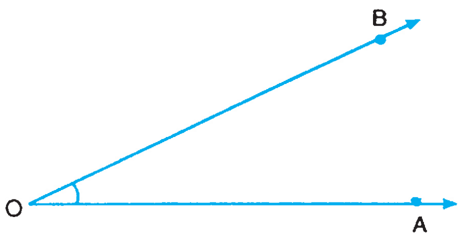


Fig. 10.11(A)

This angle may be named as angle AOB or angle BOA or simply angle O; and is written as $\angle AOB$ or $\angle BOA$ or $\angle O$. [see Fig. 10.11A]

An angle is measured in degrees. If we take any point O and draw two rays starting from it in opposite directions then the measure of this angle is taken to be 180^0 degrees, written as 180^0.

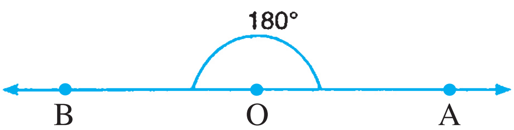


Fig. 10.12

This measure divided into 180 equal parts is called one degree (written as 1^o).

Angle obtained by two opposite rays is called a **straight angle**.

An angle of 90^0 is called a **right angle**, for example $\angle BOA$ or $\angle BOC$ is a right angle in Fig. 10.13.

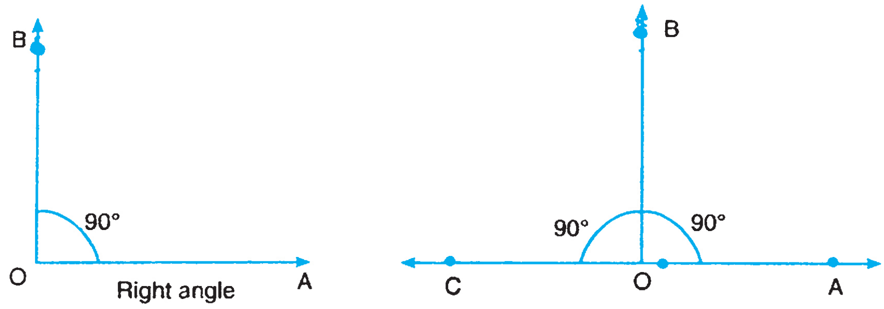


Fig. 10.13

Two lines or rays making a right angle with each other are called **perpendicular lines**. In Fig. 10.13 we can say OA is perpendicular to OB or vice-versa.

An angle less than 90^0 is called an **acute angle**. For example $\angle POQ$ is an acute angle in Fig. 10.14(a).

An angle greater than 90^0 but less than 180^0 is called an **obtuse angle**. For example, $\angle XOY$ is an obtuse angle in Fig. 10.14(b).

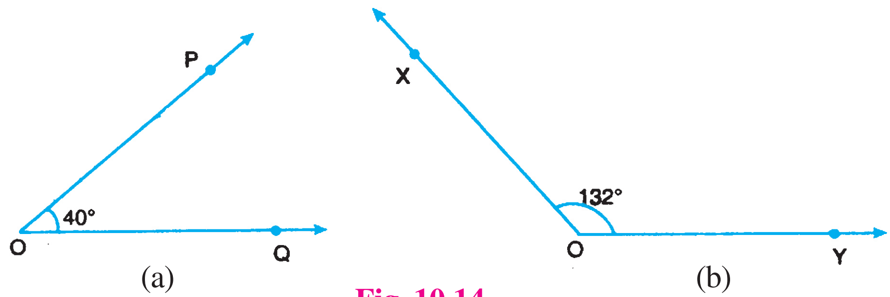


Fig. 10.14

10.2 PAIRS OF ANGLES

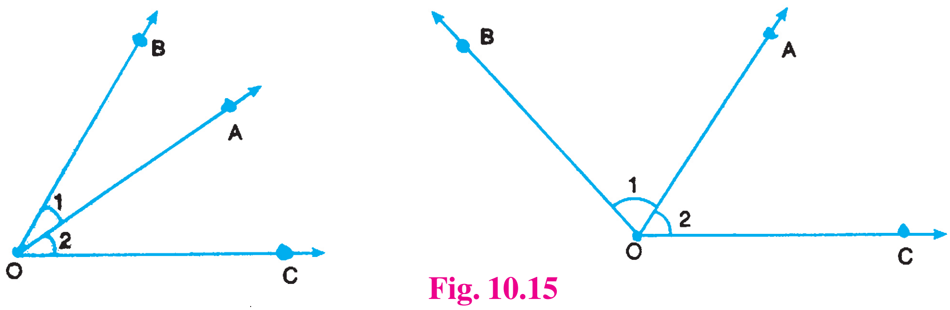


Fig. 10.15

Observe the two angles ∠1 and ∠2 in each of the figures in Fig. 10.15. Each pair has a common vertex O and a common side OA in between OB and OC. Such a pair of angles is called a '**pair of adjacent angles**'.

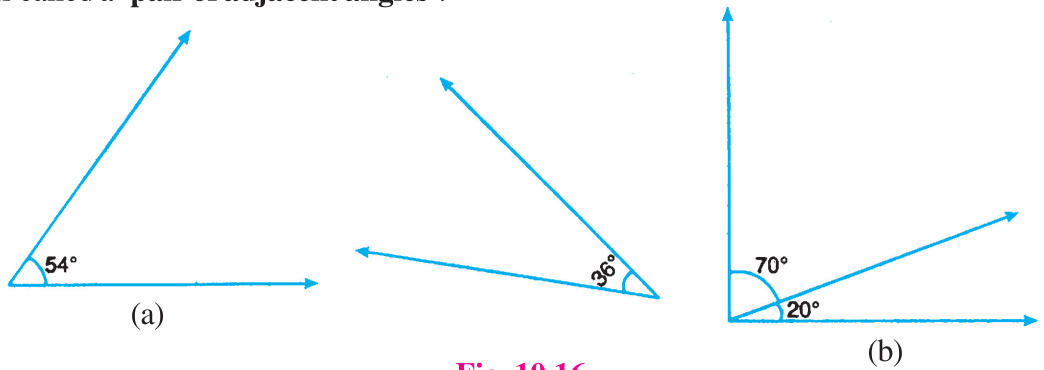


Fig. 10.16

Observe the angles in each pair in Fig. 10.16[(a) and (b)]. They add up to make a total of 90°.

A pair of angles, whose sum is 90°, is called a pair of **complementary angles**. Each angle is called the **complement** of the other.

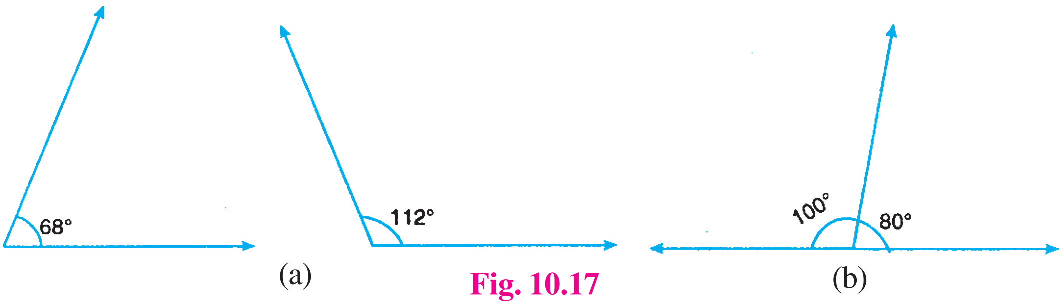


Fig. 10.17

Again observe the angles in each pair in Fig. 10.17[(a) and (b)].

These add up to make a total of 180°.

A pair of angles whose sum is 180^0, is called a pair of supplementary angles.

Each such angle is called the **supplement** of the other.

Draw a line AB. From a point C on it draw a ray CD making two angles ∠X and ∠Y.

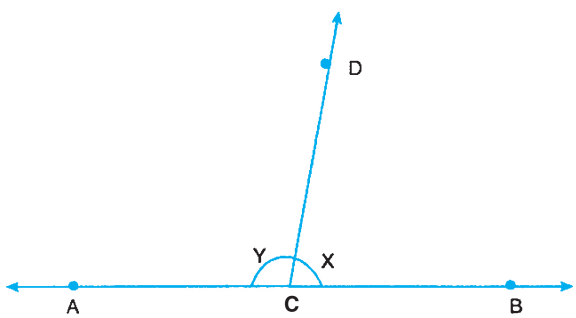


Fig. 10.18

If we measure $\angle X$ and $\angle Y$ and add, we will always find the sum to be 180^0, whatever be the position of the ray CD. We conclude

If a ray stands on a line then the sum of the two adjacent angles so formed is 180^o.

The pair of angles so formed as in Fig. 10.18 is called a **linear pair** of angles.

Note that they also make a pair of supplementary angles.

Draw two intersecting lines AB and CD, intersecting each other at O.

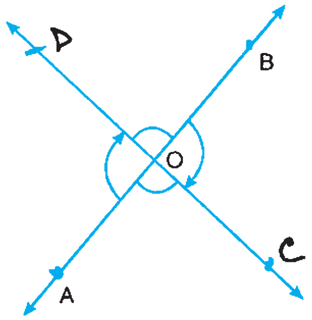


Fig. 10.19

$\angle AOC$ and $\angle DOB$ are angles opposite to each other. These make a pair of **vertically oppposite angles**. Measure them. You will always find that

$\angle AOC = \angle DOB$.

$\angle AOD$ and $\angle BOC$ is another pair of vertically opposite angles. On measuring, you will again find that

$\angle AOD = \angle BOC$

We conclude :

If two lines intersect each other, the pair of vertically opposite angles are equal.

An activity for you.

Attach two strips with a nail or a pin as shown in the figure.

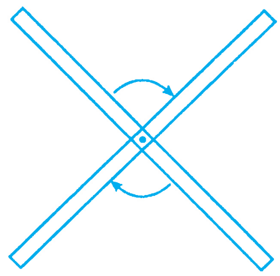


Fig. 10.20

Rotate one of the strips, keeping the other in position and observe that the pairs of vertically opposite angles thus formed are always equal.

A line which intersects two or more lines at distinct points is called a **transversal**. For example line *l* in Fig. 10.21 is a transversal.

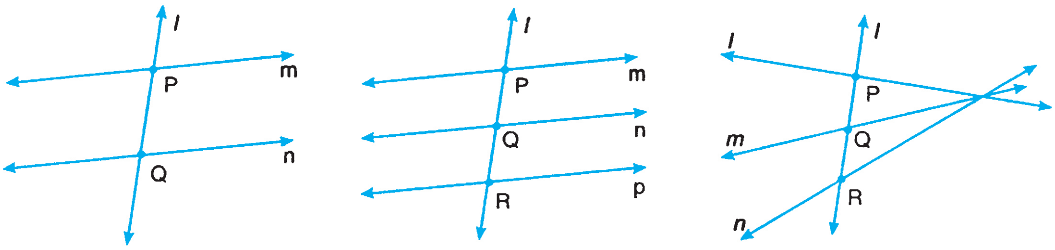


Fig. 10.21

When a transversal intersects two lines, eight angles are formed.

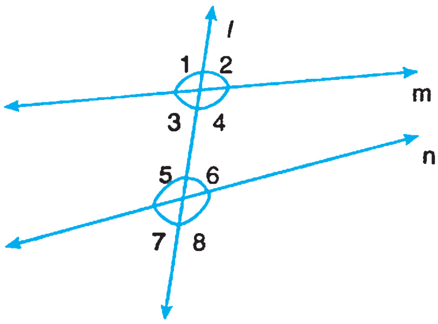


Fig. 10.22

These angles in pairs are very important in the study of properties of parallel lines. Some of the useful pairs are as follows :

(a) $\angle 1$ and $\angle 5$ is a pair of corresponding angles. $\angle 2$ and $\angle 6$, $\angle 3$ and $\angle 7$ and $\angle 4$ and $\angle 8$ are other pairs of corresponding angles.

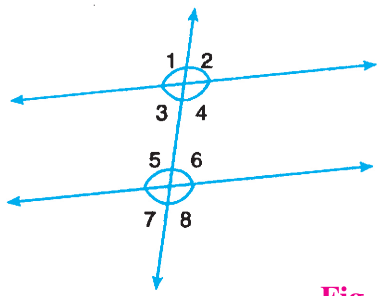
(b) $\angle 3$ and $\angle 6$ is a pair of alternate angles. $\angle 4$ and $\angle 5$ is another pair of alternate angles.

(c) $\angle 3$ and $\angle 5$ is a pair of interior angles on the same side of the transversal.

$\angle 4$ and $\angle 6$ is another pair of interior angles.

In Fig. 10.22 above, lines m and n are not parallel; as such, there may not exist any relation between the angles of any of the above pairs. However, when lines are parallel, there are some very useful relations in these pairs, which we study in the following:

When a transversal intersects two parallel lines, eight angles are formed, whatever be the position of parallel lines or the transversal.



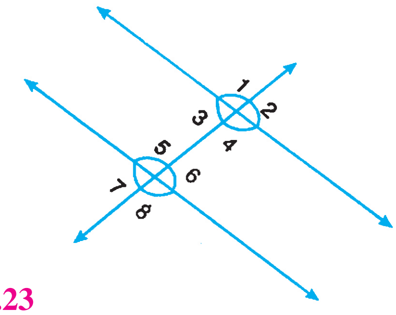


Fig. 10.23

If we measure the angles, we shall alwys find that

$$\angle 1 = \angle 5, \qquad \angle 2 = \angle 6, \qquad \angle 3 = \angle 7 \text{ and } \angle 4 = \angle 8$$

that is, angles in each pair of corresponding angles are equal.

Also $\qquad \angle 3 = \angle 6$ and $\angle 4 = \angle 5$

that is, angles in each pair of alternate angle are equal.

Also, $\qquad \angle 3 + \angle 5 = 180^\circ$ and $\angle 4 + \angle 6 = 180^\circ$.

Hence we conclude :

When a transversal intersects two parallel lines, then angles in

(i) each pair of corresponding angles are equal

(ii) each pair of alternate angles are equal

(iii) each pair of interior angles on the same side of the transversal are supplementary,

You may also verify the truth of these results by drawing a pair of parallel lines (using parallel edges of your scale) and a transversal and measuring angles in each of these pairs.

Converse of each of these results is also true. To verify the truth of the first converse, we draw a line AB and mark two points C and D on it.

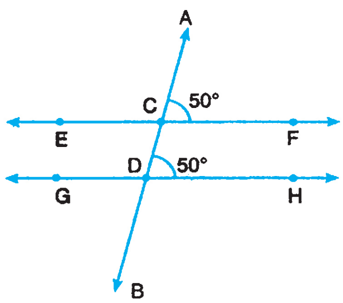


Fig. 10.24

At C and D, we construct two angles ACF and CDH equal to each other, say 50°, as shown in Fig. 10.24. On producing EF and GH on either side, we shall find that they do not intersect each other, that is, they are parallel.

In a similar way, we can verify the truth of the other two converses.

Hence we conclude that

When a transversal inersects two lines in such a way that angles in

(i) any pair of corresponding angles are equal

or (ii) any pair of alternate angles are equal

or (iii) any pair of interior angles on the same side of transversal are supplementary then the two lines are parallel.

Example 10.1 : Choose the correct answwer out of the alternative options in the following multiple choice questions.

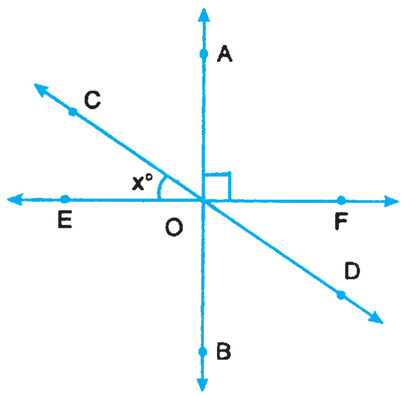


Fig. 10.25

(i) In Fig. 10.25, ∠FOD and ∠BOD are

(A) supplementary angles (B) complementary angles

(C) vertically opposite angles (D) a linear pair of angles **Ans.** (B)

(ii) In Fig. 10.25, ∠COE and ∠BOE are

(A) complementary angles (B) supplementary angles

(C) a linear pair (D) adjacent angles **Ans.** (D)

(iii) In Fig. 10.25, ∠BOD is equal to

(A) x° (B) $(90 + x)^\circ$

(C) $(90 - x)^\circ$ (D) $(180 - x)^\circ$ **Ans** (C)

(iv) An angle is 4 times its supplement; the angle is

(A) 39° (B) 72°

(C) 108° (D) 144° **Ans** (D)

(v) What value of x will make ACB a straight angle in Fig. 10.26

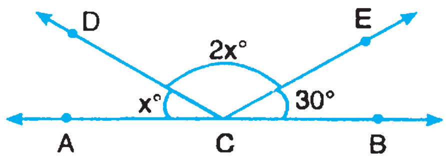


Fig. 10.26

(A) 30^o (B) 40^o

(C) 50^o (D) 60^o **Ans** (C)

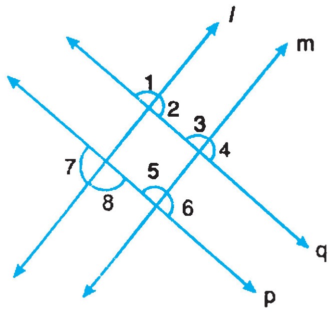


Fig. 10.27

In the above figure, l is parallel to m and p is parallel to q.

(vi) $\angle 3$ and $\angle 5$ form a pair of

(A) Alternate angles (B) interior angles

(C) vertically opposite (D) corresponding angles **Ans** (D)

(vii) In Fig. 10.27, if $\angle 1 = 80^o$, then $\angle 6$ is equal to

(A) 80^o (B) 90^o

(C) 100^o (D) 110^o **Ans** (C)

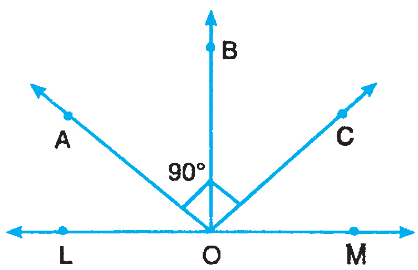


Fig. 10.28

(viii) In Fig. 10.28, OA bisects $\angle$LOB, OC bisects $\angle$MOB and $\angle AOC = 90^0$. Show that the points L, O and M are collinear.

Solution : $\angle BOL = 2\ \angle BOA$ ...(i)

and $\angle BOM = 2\ \angle BOC$ ...(ii)

Adding (i) and (ii), $\angle BOL + \angle BOM = 2\ \angle BOA + 2\angle BOC$

$\therefore\ \angle LOM = 2[\angle BOA + \angle BOC]$

$= 2 \times 90^\circ$

$= 180^\circ$ = a straight angle

$\therefore$ L, O and M are collinear.

CHECK YOUR PROGRESS 10.1.

1. Choose the correct answer out of the given alternatives in the following multiple choice questions :

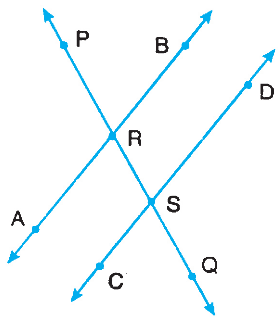


Fig. 10.29

In Fig. 10.29, AB || CD and PQ intersects them at R and S respectively.

(i) $\angle ARS$ and $\angle BRS$ form

(A) a pair of alternate angles

(B) a linear pair

(C) a pair of corresponding angles

(D) a pair of vertically opposite angles

(ii) $\angle ARS$ and $\angle RSD$ form a pair of

(A) Alternate angles (B) Vertically opposite angles

(C) Corresponding angles (D) Interior angles

(iii) If $\angle PRB = 60^\circ$, then $\angle QSC$ is

(A) 120° (B) 60°

(C) 30^{o} (D) 90^{o}

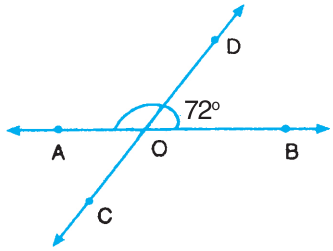


Fig. 10.30

(iv) In Fig. 10.30 above, AB and CD intersect at O. $\angle COB$ is equal to

(A) 36^{o} (B) 72^{o}

(C) 108^{o} (D) 144^{o}

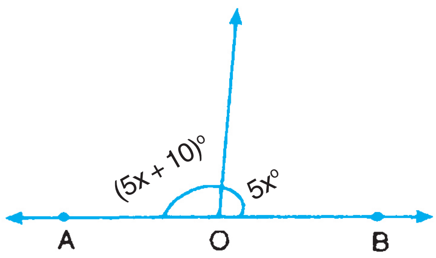


Fig. 10.31

2. In Fig. 10.31 above, AB is a straight line. Find x

3. In Fig. 10.32 below, *l* is parallel to m. Find angles 1 to 7.

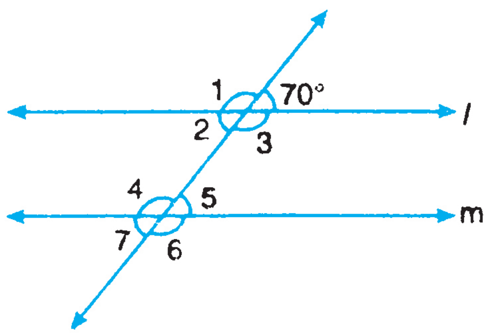


Fig. 10.32

10.3 TRIANGLE, ITS TYPES AND PROPERTIES

Triangle is the simplest polygon of all the closed figures formed in a plane by three line segments.

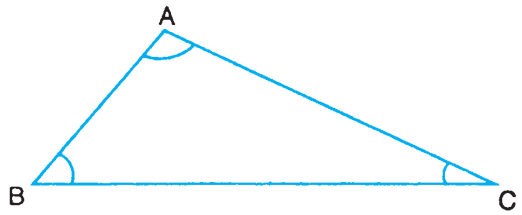


Fig. 10.33

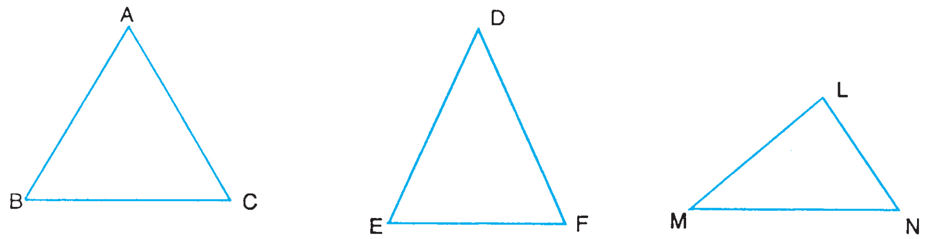
It is a closed figure formed by three line segments having six elements, namely three **angles** (i) ∠ABC or ∠B (ii) ∠ACB or ∠C (iii) ∠CAB or ∠A and three **sides** : (iv) AB (v) BC (vi) CA

It is named as Δ ABC or Δ BAC or Δ CBA and read as triangle ABC or triangle BAC or triangle CBA.

10.3.1 Types of Triangles

Triangles can be classified into different types in two ways.

(a) On the basis of sides



(i) (ii) (iii)

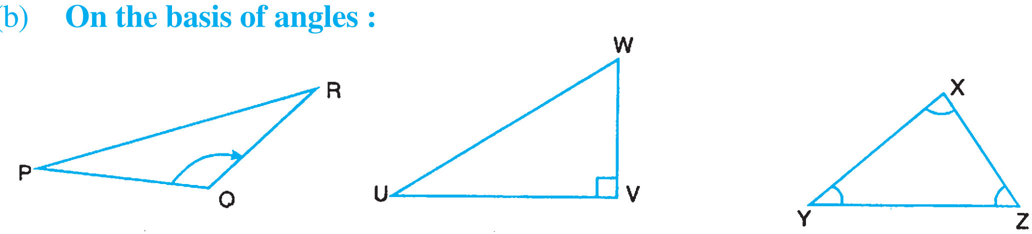
Fig. 10.34

(i) **Equilateral triangle :** a triangle in which all the three sides are equal is called an equilateral trangle. [Δ ABC in Fig. 10.34(i)]

(ii) **Isosceles triangle :** A triangle in which two sides are equal is called an isosceles triangle. [ΔDEF in Fig. 10.34(ii)]

(iii) **Scalene triangle :** A triangle in which all sides are of different lengths, is called a sclene triangle [Δ LMN in Fig. 10.34(iii)]

(b) **On the basis of angles :**



(i) (ii) (iii)

Fig. 10.35

(i) **Obtuse angled triangle :** A triangle in which one of the angles is an obtuse angle is called an **obtuse angled triangle** or simply obtuse triangle [Δ PQR is Fig. 10.35(i)]

(ii) **Right angled triangle :** A triangle in which one of the angles is a right angle is called a **right angled triangle** or right triangle. [Δ UVW in Fig. 10.35(ii)]

(iii) **Acute angled triangle :** A triangle in which all the three angles are acute is called an **acute angled triangle** or acute triangle [Δ XYZ in Fig. 10.35(iii)

Now we shall study some important properties of angles of a triangle.

10.3.2 Angle Sum Property of a Triangle

We draw two triangles and measure their angles.

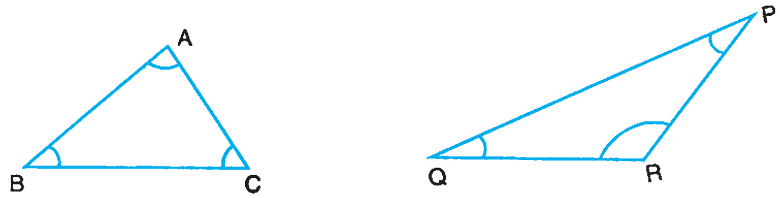


Fig. 10.36

In Fig. 10.36 (a), $\angle A = 80^o$, $\angle B = 40^o$ and $\angle C = 60^o$

$\therefore \quad \angle A + \angle B + \angle C = 80^o + 40^o + 60^o = 180^o$

In Fig. 10.36(b), $\angle P = 30^o$, $\angle Q = 40^o$, $\angle R = 110^o$

$\therefore \quad \angle P + \angle Q + \angle R = 30^o + 40^o + 110^o = 180^o$

What do you observe? Sum of the angles of triangle in each case in 180^0.

We will prove this result in a logical way naming it as a theorem.

Theorem : The sum of the three angles of triangle is 180º.

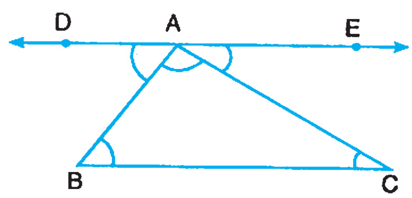


Fig. 10.37

Given : A triangle ABC

To Prove : $\angle A + \angle B + \angle C = 180^0$

Construction : Through A, draw a line DE parallel to BC.

Proof : Since DE is parallel to BC and AB is a transversal.

$\therefore \quad \angle B = \angle DAB$ (Pair of alternate angles)

Similarly $\angle C = \angle EAC$ (Pair of alternate angles)

$\therefore \angle B + \angle C = \angle DAB + \angle EAC$ ...(1)

Now adding $\angle A$ to both sides of (1)

$$\angle A + \angle B + \angle C = \angle A + \angle DAB + \angle EAC$$

$$= 180^\circ \quad \text{(Angles making a straight angle)}$$

10.3.3 Exterior Angles of a Triangle

Let us produce the side BC of ΔABC to a point D.

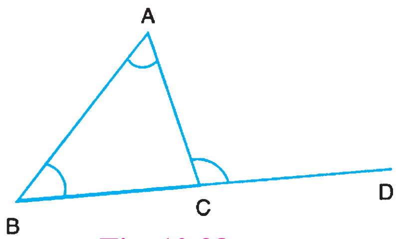


Fig. 10.38

In Fig. 10.39, observe that there are six exterior angles of the ΔABC, namely $\angle 1$, $\angle 2$, $\angle 3$, $\angle 4$, $\angle 5$ and $\angle 6$.

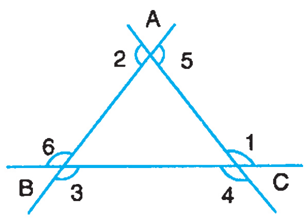


Fig. 10.39

In Fig. 10.38, $\angle ACD$ so obtained is called an exterior angle of the ΔABC. Thus,

The angle formed by a side of the triangle produced and another side of the triangle is called an exterior angle of the triangle.

Corresponding to an exterior angle of a triangle, there are two interior opposite angles.

Interior opposite angles are the angles of the triangle not forming a linear pair with the given exterior angle.

For example in Fig. 10.38, $\angle A$ and $\angle B$ are the two interior opposite angles corresponding to the exterior angle ACD of ΔABC. We measure these angles.

$$\angle A = 60^\circ$$

$$\angle B = 50^\circ$$

and $\angle ACD = 110^o$

We observe that $\angle ACD = \angle A + \angle B$.

This observation is true in general.

Thus, we may conclude :

An exterior angle of a triangle is equal to the sum of the two interior opposite angles.

Examples 10.3 : Choose the correct answer out of the given alternatives in the following multiple choice questions:

(i) Which of the following can be the angles of a triangle?

(A) 65^o, 45^o and 80^o (B) 90^o, 30^o and 61^o

(C) 60^o, 60^o and 59^o (D) 60^o, 60^o and 60^o. **Ans** (D)

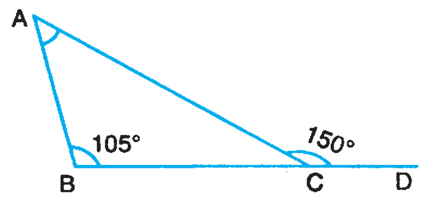


Fig. 10.40

(ii) In Fig. 10.40 $\angle A$ is equal to

(A) 30^o (B) 35^o

(C) 45^o (D) 75^o **Ans** (C)

(iii) In a triangle, one angle is twice the other and the third angle is 60^0. Then the largest angle is

(A) 60^o (B) 80^o

(C) 100^o (D) 120^o **Ans** (B)

Example 10.4:

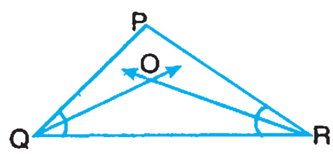


Fig. 10.41

In Fig. 10.41, bisctors of $\angle PQR$ and $\angle PRQ$ intersect each other at O. Prove that $\angle QOR = 90^o + \frac{1}{2}\angle P$.

Solution : $\angle QOR = 180^o - \frac{1}{2}[\angle PQR + \angle PRQ)]$

$$= 180^o - \frac{1}{2}(\angle PQR + \angle PRQ)$$

$$= 180^o - \frac{1}{2}(180^o \ \ - \angle P)$$

$$= 180^o - 90^o + \frac{1}{2}\angle P = 90^o + \frac{1}{2}\angle P$$

CHECK YOUR PROGRESS 10.2

1. Choose the correct answer out of given alternatives in the following multiple choice questions:

   (i) A triangle can have

   (A) Two right angles (B) Two obtuse angles

   (C) At the most two acute angles (D) All three acute angles

   (ii) In a right triangle, one exterior angles is 120^0, The smallest angle of the triangles is

   (A) 20^o (B) 30^0

   (C) 40^o (D) 60^0

   (iii)

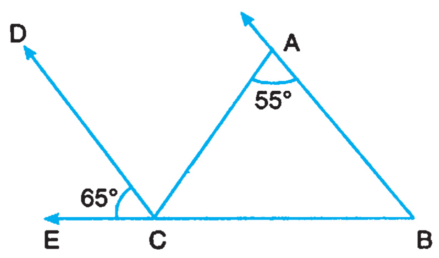


Fig. 10.42

   In Fig. 10.42, CD is parallel to BA. $\angle ACB$ is equal to

   (A) 55^o (B) 60^o

   (C) 65^o (D) 70^o

2. The angles of a triangle are in the ratio 2 : 3 : 5, find the three angles.

3. Prove that the sum of the four angles of a quadrilateral is 360^o.

4. In Fig. 10.43, ABCD is a trapezium such that AB||DC. Find ∠D and ∠C and verify that sum of the four angles is 360°.

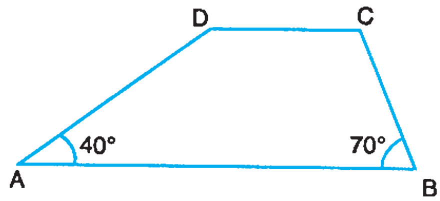


Fig. 10.43

5. Prove that if one angle of a triangle is equal to the sum of the other two angles, then it is a right triangle.

6. In Fig. 10.44, ABC is triangle such that ∠ABC = ∠ACB. Find the angles of the triangle.

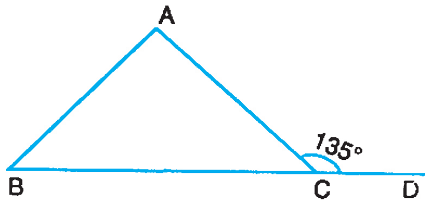


Fig. 10.44

10.4 LOCUS

During the game of cricket, when a player hits the ball, it describes a path, before being caught or touching the ground.

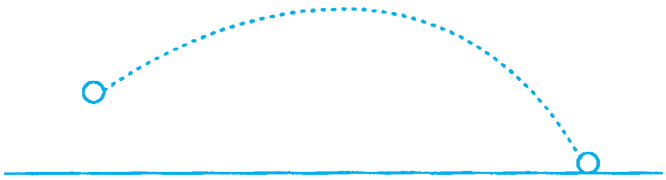


Fig. 10.44

The path described is called Locus.

A figure in geometry is a result of the path traced by a point (or a very small particle) moving under certain conditions.

For example:

(1) Given two parallel lines *l* and m, also a point P between them equidistant from both the lines.

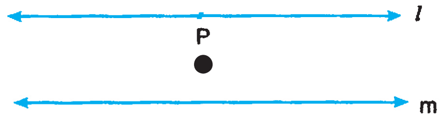


Fig. 10.45

If the particle moves so that it is equidistant from both the lines, what will be its path?

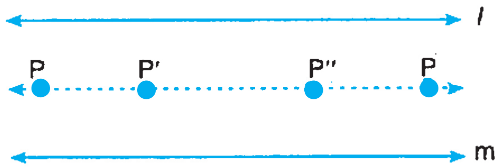


Fig. 10.46

The path traced by P will be a line parallel to both the lines and exactly in the middle of them as in Fig. 10.46.

(2) Given a fixed point O and a point P at a fixed distance *d*.

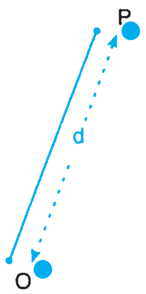


Fig. 10.47

If the point P moves in a plane so that it is always at a constant distance *d* from the fixed point O, what will be its path?

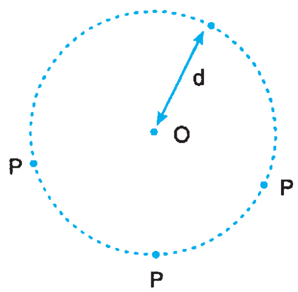


Fig. 10.48

The path of the moving point P will be a circle as shown in Fig. 10.48.

(3) Place a small piece of chalk stick or a pebble on top of a table. Strike it hard with a pencil or a stick so that it leaves the table with a certain speed and observe its path after it leaves the table.

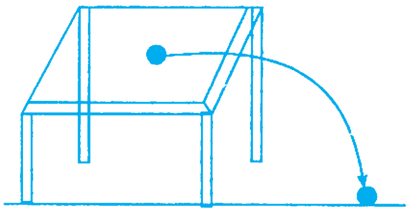


Fig. 10.49

The path traced by the pebble will be a curve (part of what is known as a parabola) as shown in Fig. 10.49.

Thus, locus of a point moving under certain conditions is the path or the geometrical figure, every point of which satisfies the given conditon(s).

10.4.1 Locus of a point equidistant from two given points

Let A and B be the two given points.

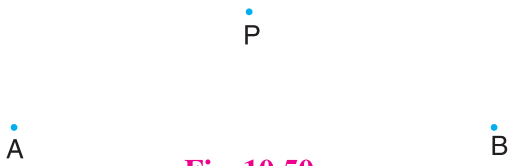


Fig. 10.50

We have to find the locus of a point P such that PA = PB.

Joint AB. Mark the mind point of AB as M. Clearly, M is a point which is equidistant from A and B. Mark another point P using compasses such that PA = PB. Join PM and extend it on both sides. Using a pair of divider or a scale, it can easily be verified that every point on PM is equidistant from the points A and B. Also, if we take any other point Q not lying on line PM, then $QA \neq QB$.

Also $\angle AMP = \angle BMP = 90^{o}$

That is, PM is the perpendicular bisector of AB.

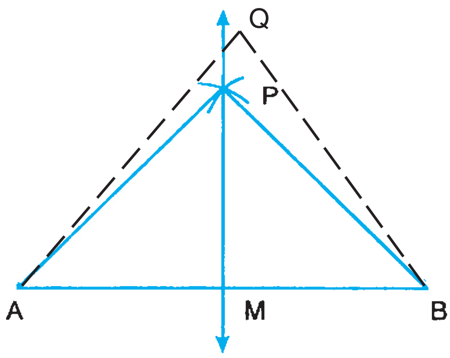


Fig. 10.51

Thus, we may conclude the following:

The locus of a point equidistant from two given poitns is the perpendicular bisector of the line segment joining the two points.

Activity for you :

Mark two points A and B on a sheet of paper and join them. Fold the paper along mid-point of AB so that A coincides with B. Make a crease along the line of fold. This crease is a straight line. This is the locus of the point equidistant from the given points A and B. It can be easily checked that very point on it is equidistant from A and B.

10.4.2 Locus of a point equidistant from two lines intersecting at O

Let AB and CD be two given lines intersecting at O.

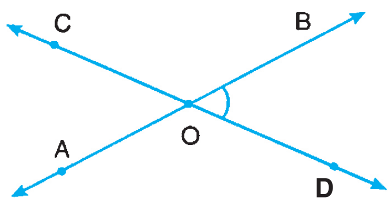


Fig. 10.52

We have to find the locus of a point P which is equidistant from both AB and CD.

Draw bisectors of $\angle BOD$ and $\angle BOC$.

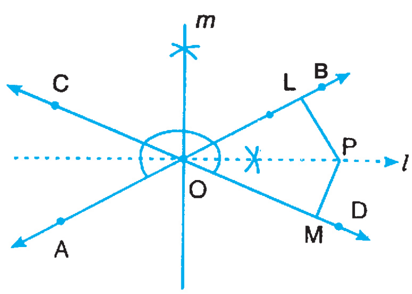


Fig. 10.53

If we take any point P on any bisector l or m, we will find perpendicular distances PL and PM of P from the lines AB and CD are equal.

that is, $\quad PL = PM$

If we take any other point, say Q, not lying on any bisector l or m, then QL will not be equal to QM.

Thus, we may conclude :

The locus of a point equidistant from two intersecting lines is the pair of lines, bisecting the angles formed by the given lines.

Activity for you :

Draw two lines AB and CD intersecting at O, on a sheet of paper. Fold the paper through O so that AO falls on CO and OD falls on OB and mark the crease along the fold. Take a piont P on this crease which is the bisector of ∠BOD and check using a set square that

PL = PM

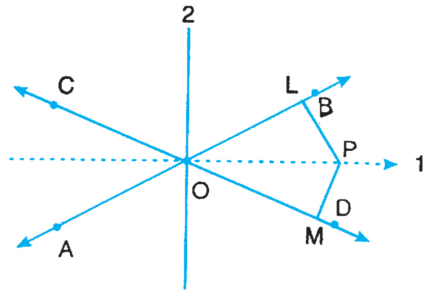


Fig. 10.54

In a similar way find the other bisector by folding again and getting crease 2. Any point on this crease 2 is also equidistant from both the lines.

Example 10.5 : Find the locus of the centre of a circle passing through two given points.

Solution : Let the two given points be A and B. We have to find the position or positions of centre O of a circle passing through A and B.

Fig. 10.55

Point O must be equidistant from both the points A and B. As we have already learnt, the locus of the point O will be the perpendicular bisector of AB.

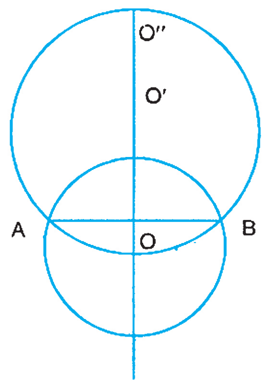


Fig. 10.56

CHECK YOU PROGRESS 10.3

1. Find the locus of the centre of a circle passing through three given points A, B and C which are non-collinear.

2. There are two villages certain distance apart. A well is to be dug so that it is equidistant from the two villages such that its distance from each village is not more than the distance between the two villages. Representing the villages by points A and B and the well by point P. show in a diagram the locus of the point P.

3. Two straight roads AB and CD are intersecting at a point O. An observation post is to be constructred at a distance of 1 km from O and equidistant from the roads AB and CD. Show in a diagram the possible locations of the post.

4. Find the locus of a point which is always at a distance 5 cm from a given line AB.

LET US SUM UP

- A line extends to inifinity on both sides and a line segment is only a part of it between two points.
- Two distinct lines in a plane may either be intersecting or parallel.
- If three or more lines intersect in one point only then they are called cocurrent lines.
- Two rays starting from a common point form an angle.
- A pair of angles, whose sum is 90^0 is called a pair of complementary angles.
- A pair of angles whose sum is 180^0 is called a pair of supplementary angles.
- If a ray stands on a line then the sum of the two adjacent angles, so formed is 180^0
- If two lines intersect each other the pairs of vertically opposite angles are equal
- When a transversal intersects two parallel lines, then

  (i) corresponding angles in a pair are equal.

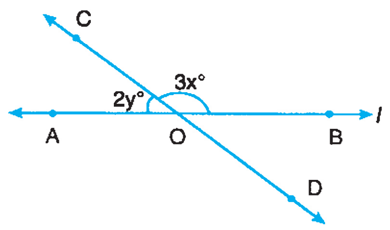
  (ii) alternate angles are equal.

  (iii) interior angles on the same side of the transversal are supplementary.
- The sum of the angles of a triangle is 180^0
- An exterior angle of a triangle to equal to the sum of the two interior opposite angles
- Locus of a point equidistant from two given points is the perpendicular bisector of the line segment joing the points.

- The locus of a point equidistant from the intersecting lines is the pair of lines, bisecting the angle formed by the given lines.

TERMINAL EXERCISE

1. In Fig. 10.57, if x = 42, then determine (a) y (b) ∠AOD



2.

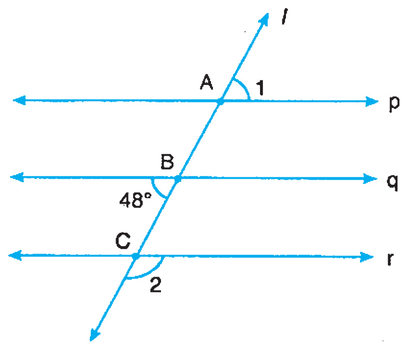


Fig. 10.58

In the above figure p, q and r are parallel lines intersected by a transversal l at A, B and C respectively. Find ∠1 and ∠2.

3. The sum of two angles of a triangle is equal to its third angle. Find the third angle. What type of triangle is it?

4.

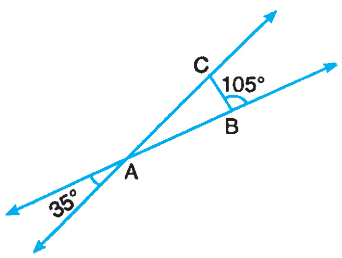


Fig. 10.59

In Fig. 10.59, sides of Δ ABC have been produced as shown. Find the angles of the triangle.

5.

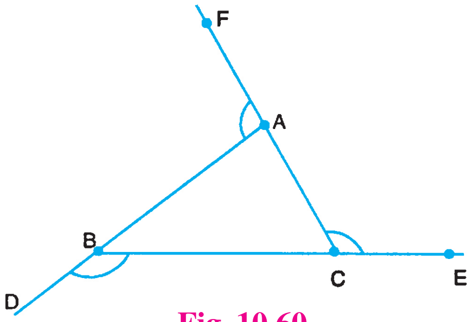


Fig. 10.60

In Fig. 10.60, sides AB, BC and CA of the triangle ABC have been produced as shown. Show that the sum of the exterior angles so formed is 360º.

6.

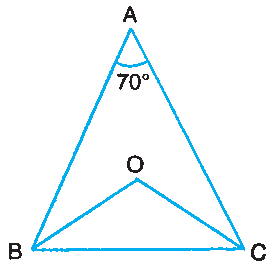


Fig. 10.61

In Fig. 10.61 ABC is a triangle in which bisectors of $\angle B$ and $\angle C$ meet at O. Show that $\angle BOC = 125^{o}$.

7.

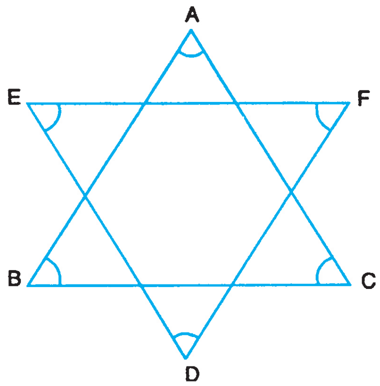


Fig. 10.62

In Fig. 10.62 above, find the sum of the angles, $\angle A$, $\angle F$, $\angle C$, $\angle D$, $\angle B$ and $\angle E$.

8.

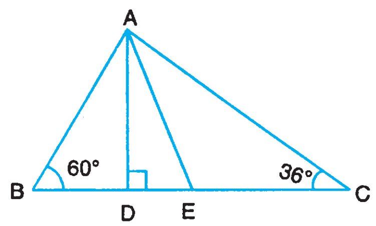


Fig. 10.63

In Fig. 10.63 in Δ ABC, AD is perpendicular to BC and AE is bisector of ∠BAC. Find ∠DAE,

9.

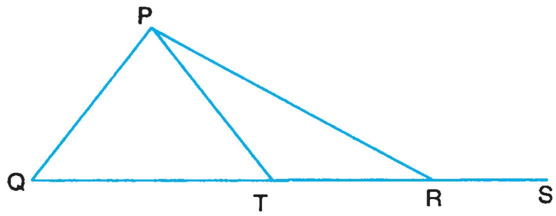


Fig. 10.64

In Fig. 10.64 above, in Δ PQR, PT is bisector of ∠P and QR is produced to S. Show that ∠PQR + ∠PRS = 2 ∠PTR.

10. Prove that the sum of the (interior) angles of a pentagon is 540^0.

11. Find the locus of a point equidistant from two parallel lines *l* and m at a distance of 5 cm from each other.

12. Find the locus of a point equidistant from points A and B and also equidistant from rays AB and AC of Fig. 10.65.

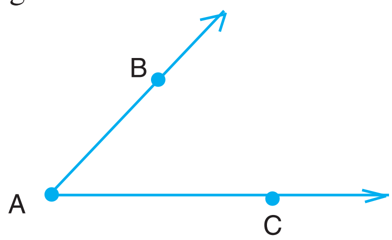


Fig. 10.65

ANSWERS TO CHECK YOUR PROGRESS

10.1

1. (i) (B)   (ii) (A)   (iii) (B)   (iv) (C)

2. $x = 17^0$.

3. $\angle 1 = \angle 3 = \angle 4 = \angle 6 = 110^o$

and $\angle 2 = \angle 5 = \angle 7 = 70^o$.

10.2

1. (i) (D)   (ii) (B)   (iii) (B)

2. 36^o, 54^o and 90^o 4. $\angle D = 140^o$ and $\angle C = 110^o$

6. $\angle ABC = 45^o$, $\angle ACB = 45^o$ and $\angle A = 90^o$

10.3

1. Only a point, which is the point of intersection of perpendicular bisectors of AB and BC.

2. Let the villages be A and B, then locus will be the line segment PQ, perpendicular bisector of AB such that

   AP = BP = QA = QB = AB

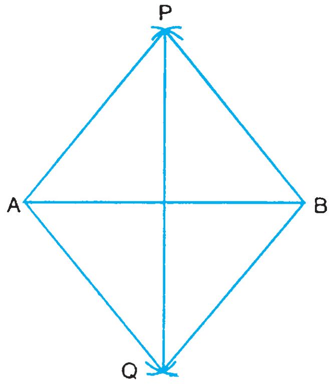


Fig. 10.65

3. Possible locations will be four points two points P and Q on the bisector of ∠AOC and two points R and S on the bisector of ∠BOC.

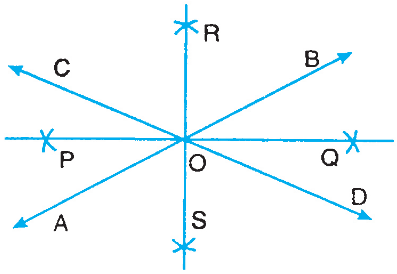


Fig. 10.66

4. Two on either side of AB and lines parallel to AB at a distance of 5 cm from AB.

ANSWERS TO TERMINAL EXERCISE

1. (a) y = 27 (b) = 126°
2. ∠1 = 48° and ∠2 = 132°
3. Third angle = 90°, Right triangle
4. ∠A = 35°, ∠B = 75° ∠C = 70°
7. 360°
8. 12°
11. A line parallel to locus *l* and m at a distance of 2.5 cm from each.
12. Point of intersection of the perpendicular bisector of AB and bisector of ∠BAC.

11

CONGRUENCE OF TRIANGLES

You might have observed that leaves of different trees have different shapes, but leaves of the same tree have almost the same shape. Although they may differ in size. The geometrical figures which have same shape and same size are called congruent figures and the property is called congruency.

In this lesson you will study congruence of two triangles, some relations between their sides and angles in details.

OBJECTIVES

After studying this lesson, you will be able to

- *verify and explain whether two given figures are congruent or not.*
- *state the criteria for congruency of two triangles and apply them in solving problems.*
- *prove that angles opposite to equal sides of a triangle are equal.*
- *prove that sides opposite to equal angles of a triangle are equal.*
- *prove that if two sides of triangle are unequal, then the longer side has the greater angle opposite to it.*
- *state and verify inequalities in a triangle involving sides and angles.*
- *solve problems based on the above results.*

EXPECTED BACKGROUND KNOWLEDGE

- Recognition of plane geometrical figures
- Equality of lines and angles
- Types of angles
- Angle sum property of a triangle
- Paper cutting and folding.

11.1 CONCEPT OF CONGRUENCE

In our daily life you observe various figures and objects. These figures or objects can be categorised in terms of their shapes and sizes in the following manner.

(i) Figures, which have different shapes and sizes as shown in Fig. 11.1

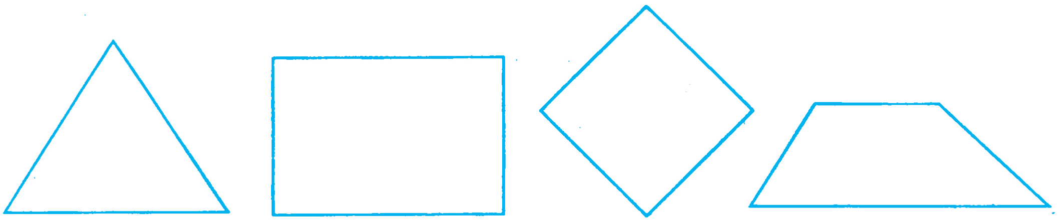


Fig. 11.1

(ii) Objcts, which have same shpaes but different sizes as shown in Fig. 11.2

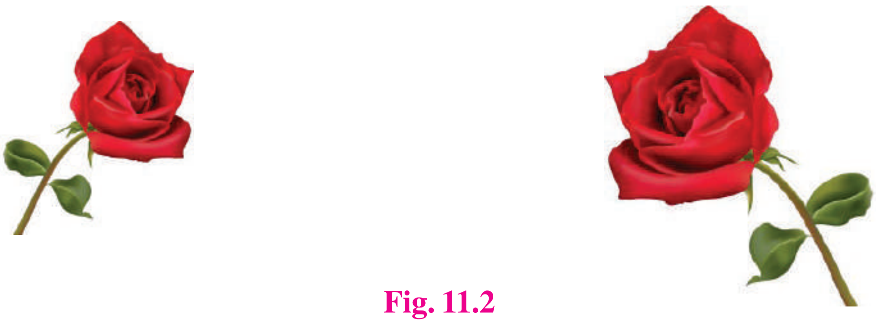


Fig. 11.2

(iii) Two one-rupee coins.

Fig. 11.3

(iv) Two postage stamps on post cards

Fig. 11.4

(v) Two photo prints of same size from the same negative.

Fig. 11.5

We will deal with the figures which have same shapes and same sizes.

Two figures, which have the same shape and same size are called congruent figures and this property is called congruence.

11.1.1. Activity

Take a sheet of paper, fold it in the middle and keep a carbon (paper) between the two folds. Now draw a figure of a leaf or a flower or any object which you like, on the upper part of the sheet. You will get a carbon copy of it on the sheet below.

The figure you drew and its carbon copy are of the same shape and same size. Thus, these are congruent figures. Observe a butterfly folding its two wings. These appear to be one.

11.1.2 Criteria for Congruence of Some Figures

Congruent figures, when palced one over another, exactly coincide with one another or cover each other. In other words, two figures will be congruent, if parts of one figure are equal to the corresponding parts of the other. For example :

(1) Two line - segments are congruent, when they are of equal length.

Fig. 11.6

(2) Two squares are congruent if their sides are equal.

Fig. 11.7

(3) Two circles are congruent, if their radii are equal, implying their circumferences are also equal.

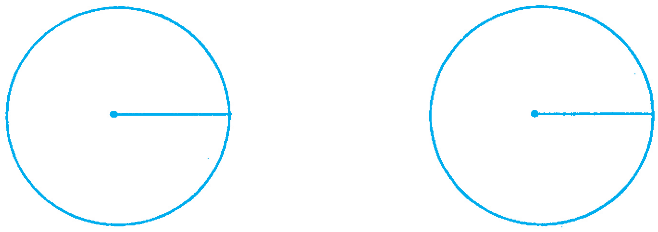


Fig. 11.8

11.2 CONGRUENCE OF TRIANGLES

Triangle is a basic rectilinear figure in geometry, having minimum number of sides. As such congruence of triangles plays a very important role in proving many useful results. Hence this needs a detailed study.

Two triangles are congruent, if all the sides and all the angles of one are equal to the corresponding sides and angles of other.

For example, in triangles PQR and XYZ in Fig. 11.9

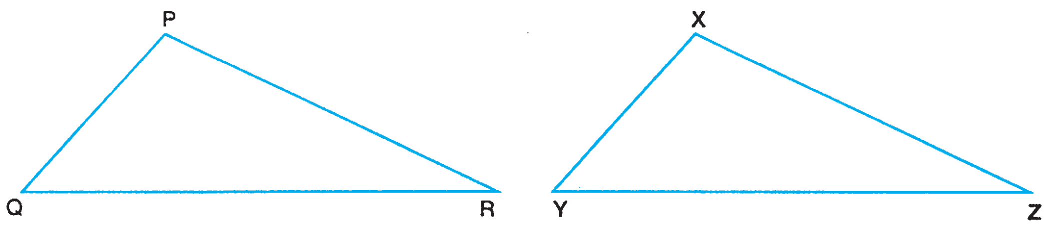


Fig. 11.9

$$PQ = XY, PR = XZ, QR = YZ$$

$$\angle P = \angle X, \angle Q = \angle Y \text{ and } \angle R = \angle Z$$

Thus we can say

Δ PQR is congruent to Δ XYZ and we write

$$\Delta PQR \cong \Delta XYZ$$

Relation of congruence between two triangles is always written with corresponding or matching parts in proper order.

Here $\Delta PQR \cong \Delta XYZ$

also means P corresponds to X, Q corresponds to Y and R corresponds to Z.

This congruence may also be written as $\Delta\ QRP \cong \Delta\ YZX$ whichmeans, Q corresponds to Y, R corresponds to Z and P corresponds to X. It also means corresponding parts, (elements) are equal, namely

$$QR = YZ,\ RP = ZX,\ QP = YX,\ \angle Q = \angle Y,\ \angle R = \angle Z$$

and $\angle P = \angle X$

This congruence may also be written as

$$\Delta\ RPQ \cong \Delta\ ZXY$$

but NOT as $\Delta\ PQR \cong \Delta\ YZX.$

Or NOT as $\Delta\ PQR \cong \Delta\ ZXY.$

11. 3 CRITERIA FOR CONGRUENCE OF TRIANGLES

In order to prove, whether two triangles are congruent or not, we need to know that all the six parts of one triangle are equal to the corresponding parts of the other triangle. We shall now learn that it is possible to prove the congruence of two triangles, even if we are able to know the equality of three of their corresponding parts.

Consider the triangle ABC in Fig. 11.10

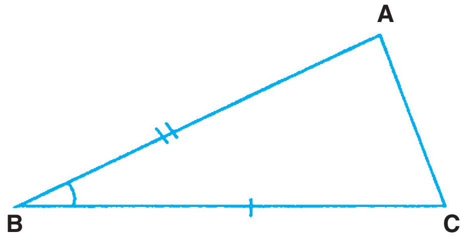


Fig. 11.10

Construct another triangle PQR such that $QR = BC$, $\angle Q = \angle B$ and $PQ = AB$. (See Fig. 11.11)

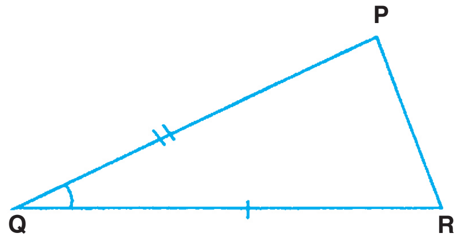


Fig. 11.11

If we trace or cut out triangle ABC and place it over triangle PQR. we will observe that one covers the other exactly. Thus, we may say that they are congruent.

Alternatively we can also measure the remaining parts, and observe that

$AC = PR, \angle A = \angle P$ and $\angle C = \angle R$

showing that $\Delta\ PQR \cong \Delta\ ABC$.

It should be noted here that in constructing Δ PQR congruent to Δ ABC we used only two parts of sides PQ = AB, QR = BC and the included angle between them $\angle Q = \angle B$.

This means that equality of these three corresponding parts results in congruent triangles. Thus we have

Criterion 1 : If any two sides and the included angle of one triangle are equal to the corresponding sides and the included angle of the other triangle, the two triangles are congruent.

This criterion is referred to as SAS (Side Angle Side).

Again, consider Δ ABC in Fig. 11.12

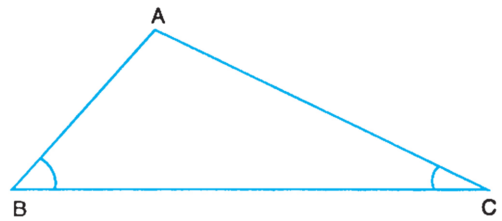


Fig. 11.12

Construct another Δ PQR such that, QR = BC, $\angle Q = \angle B$ and $\angle R = \angle C$. (See Fig. 11.13)

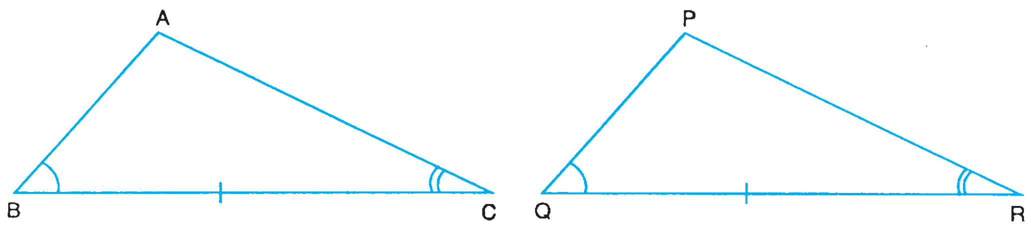


Fig. 11.13

By superimposition or by measuring the remaining corresponding parts, we observe that $\angle P = \angle A$, PQ = AB and PR = AC establishing that Δ PQR $\cong$ Δ ABC, which again means that equality of the three corresponding parts (two angles and the inluded side) of two triangles results in congruent triangles.

We also know that the sum of the three angles of a triangle is 180°, as such if two angles of one triangle are equal to the corresponding angles of another triangle, then the third angles will also be equal. Thus instead of included side we may have any pair of corresponding sides equal. Thus we have

Criterion 2 : If any two angles and one side of a triangle are equal to corresponding angles and the side of another triangle, then the two triangles are congruent.

This criterion is referred to as ASA or AAS (Angle Side Angle or Angle Angle Side)

11.3.1 Activity

In order to explore another criterion we again take a triangle ABC (See Fig. 11.14)

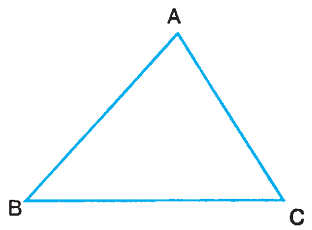


Fig. 11.14

Now take three thin sticks equal in lengths to sides AB, BC and CA of Δ ABC. Place them in any order to form Δ PQR or Δ P′Q′R′ near the Δ ABC (Fig. 11.15)

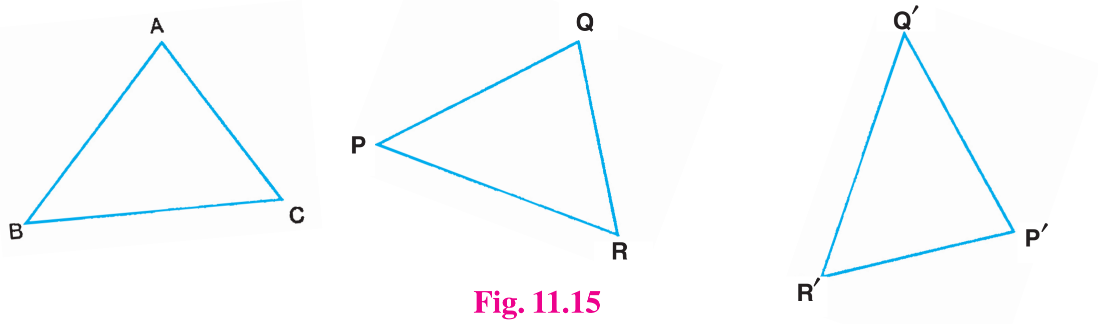


Fig. 11.15

By measuring the corresponding angles. we find that, $\angle P = \angle P' = \angle A$, $\angle Q = \angle Q' = \angle B$ and $\angle R = \angle R' = \angle C$, establishing that

$$\Delta\ PQR \cong \Delta\ P'Q'R' \cong \Delta\ ABC$$

which means that equality of the three corresponding sides of two triangles results in congruent triangles. Thus we have

Criterion 3 : If the three sides of one trianle are equal to the corresponding sides of another triangle, then the two triangles are congruent.

This is referred to as SSS (Side, Side, Side), criterion.

Similarly, we can establish one more criterion which will be applicable for two right trangles only.

Criterion 4 : If the hypotenuse and a side of one right triangle are respectively equal to the hypotenuse and a side of another right triangle, then the two triangles are congruent.

This criterion is referred to as RHS (Right Angle Hypotenuse Side).

Using these criteria we can easily prove, knowing three corresponding parts only, whether two triangles are congruent and establish the equality of remaining corresponding parts.

Example 11.1 : In which of the following criteria, two given triangles are **NOT** congruent.

(a) All corresponding sides are equal

(b) All corresponding angles are equal

(c) All corresponding sides and their included angles are equal

(d) All corresponding angles and any pair of corresponding sides are equal.

Ans. (b)

Example 11.2 : Two rectilinear figures are congruent if they have

(a) All corresponding sides equal

(b) All corresponding angles equal

(c) The same area

(d) All corresponding angles and all corresponding sides equal.

Ans. (d)

Example 11.3 : In Fig. 11.16, PX and QY are perpendicular to PQ and PX = QY. Show that AX = AY.

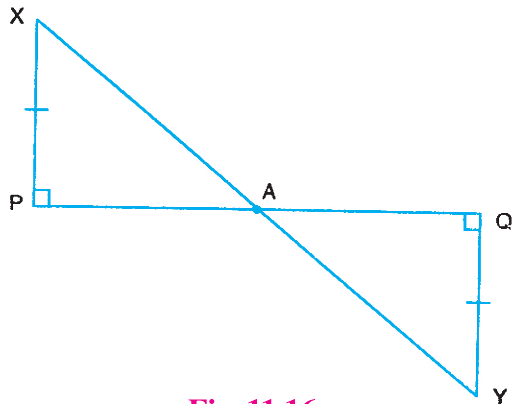


Fig. 11.16

Solution :

In Δ PAX and Δ QAY,

$\angle XPA = \angle YQA$ (Each is 90^o)

$\angle PAX = \angle QAY$ (Vertically opposite angles) and $\quad PX = QY$

$\therefore \Delta PAX \cong \Delta QAY$ (AAS)

$\therefore AX = AY.$

Example 11.4 : In Fig. 11.17, ΔABC is right triangle in which $\angle B = 90^0$ and D is the mid point of AC.

Prove that $BD = \frac{1}{2} AC.$

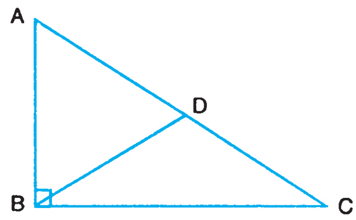


Fig. 11.17

Solution : Produce BD to E such that BD = DE. Join CE

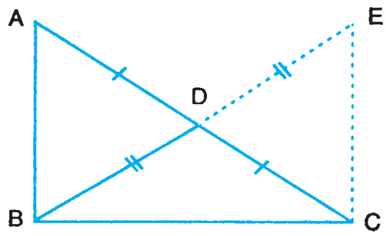


Fig.. 11.18

In ΔADB and ΔCDE,

$AD = CD$ (D being and point of AC)

$DB = DE$ (By construction)

and $\quad \angle ADB = \angle CDE$ (Vertically opposite angles)

$\therefore \quad \Delta ADB \cong \Delta CDE$ (i)

$\therefore \quad AB = EC$

Also $\quad \angle DAB = \angle DCE$

But they make a pair of alternate angles

$\therefore$ AB is parallel to EC

$\therefore \quad \angle ABC + \angle ECB = 180^0$ (Pair of interior angles)

$\therefore \quad \angle 90^0 + \angle ECB = 180^0$

$\therefore \quad \angle ECB = 180^0 - 90^0 = 90^0$

Now in Δ ABC and Δ ECB,

	AB = EC	(From (i) above)
	BC = BC	(Common)
and	$\angle ABC = \angle ECB$	(Each 90^0)
$\therefore$	$\Delta ABC \cong \Delta ECB$	
$\therefore$	AC = EB	
But	$BD = \frac{1}{2} EB$	
$\therefore$	$BD = \frac{1}{2} AC$	

CHECK YOUR PROGRESS 11.1

1. In Δ ABC (Fig. 11.19) if $\angle B = \angle C$ and AD $\perp$ BC, then $\Delta ABD \cong \Delta ACD$ by the criterion.

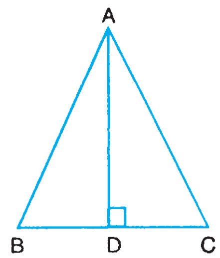


Fig. 11.19

(a) RHS (b) ASA

(c) SAS (d) SSS

2. In Fig. 11.20, $\Delta ABC \cong \Delta PQR$. This congruence may also be written as

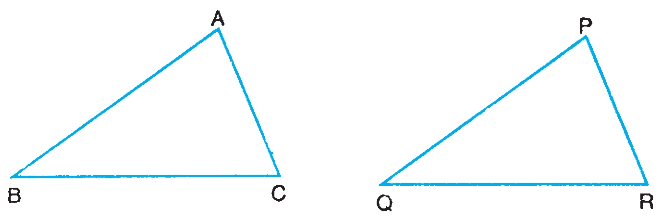


Fig. 11.20

(a) $\Delta\, BAC \cong \Delta\, RPQ$ (b) $\Delta\, BAC \cong \Delta\, QPR$

(c) $\Delta\, BAC \cong \Delta\, RQP$ (d) $\Delta\, BAC \cong \Delta\, PRQ$.

3. In order that two given triangles are congruent, along with equality of two corresponding angles we must know the equality of :

(a) No corresponding side

(b) Minimum one corresponding side

(c) Minimum two corresponding sides

(d) All the three corresponding sides

4. Two triangles are congruent if ....

(a) All three corresponding angles are equal

(b) Two angles and a side of one are equal to two angles and a side of the other.

(c) Two angles and a side of one are equal to two angles and the corresponding side of the other.

(d) One angle and two sides of one are equal to one angle and two sides of the other.

5. In Fig. 11.21, $\angle B = \angle C$ and $AB = AC$. Prove that $\Delta\, ABE \cong \Delta\, ACD$. Hence show that $CD = BE$.

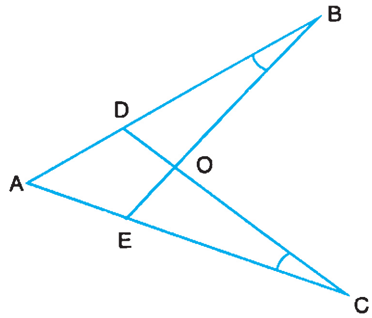


Fig. 11.21

6. In Fig. 11.22, AB is parallel to CD. If O is the mid-point of BC, show that it is also the mid-point of AD.

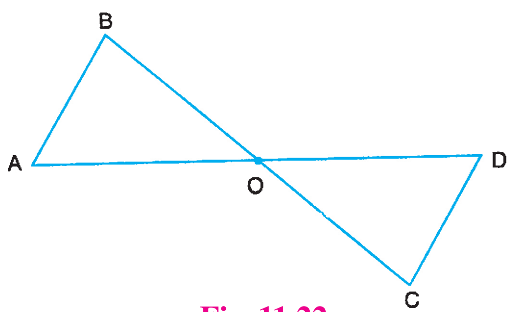


Fig. 11.22

7. In Δ ABC (Fig. 11.23), AD is ⊥ BC, BE is ⊥ AC and AD = BE. Prove that AE = BD.

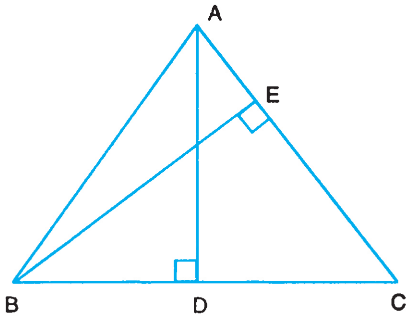


Fig. 11.23

8. From Fig. 11.24, show that the triangles are congruent and make pairs of equal angles.

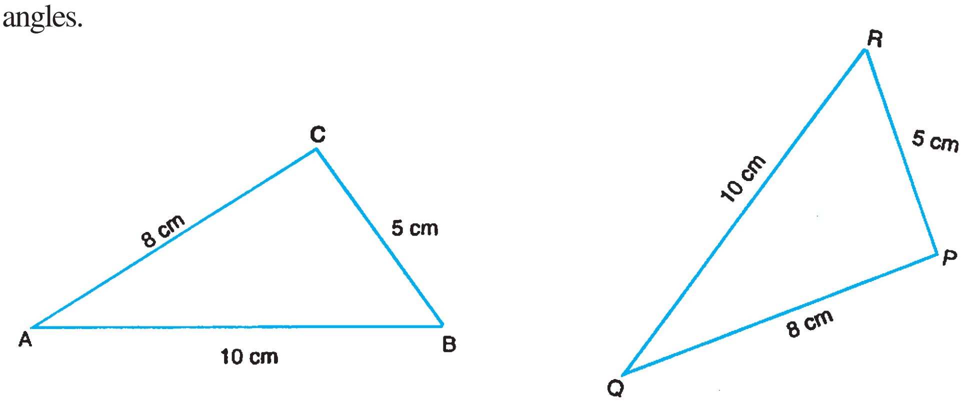


Fig. 11.24

11.4 ANGLES OPPOSITE TO EQUAL SIDES OF A TRIANGLE AND VICE VERSA

Using the criteria for congruence of triangles, we shall now prove some important theorems.

Theorem : The angles opposite to equal sides of a triangle are equal.

Given : A triangle ABC in which AB = AC.

To prove : ∠B = ∠C.

Construction : Draw bisector of ∠BAC meeting BC at D.

Proof : In Δ ABD and Δ ACD,

AB = AC (Given)

∠BAD = ∠CAD (By construction)

and AD = AD (Common)

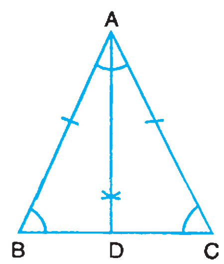


Fig. 11.25

$\Delta ABD \cong \Delta ACD$ (SAS)

Hence $\angle B = \angle C$ (Corresponding parts of congruent triangles)

The converse of the above theorem is also true. We prove it as a theorem.

11.4.1 The sides opposite to equal angles of a triangle are equal

Given : A triangle ABC in which $\angle B = \angle C$

To prove : AB = AC

Construction : Draw bisector of $\angle BAC$ meeting BC at D.

Proof : In Δ ABD and Δ ACD,

$\angle B = \angle C$ (Given)

$\angle BAD = \angle CAD$ (By construction)

and AD = AD (Common)

$\Delta ABD \cong \Delta ACD$ (SAS)

Hence AB = AC (c.p.c.t)

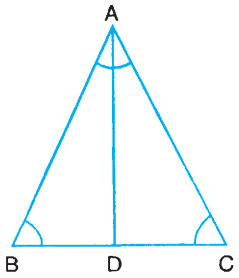


Fig. 11.26

Hence the theorem.

Example 11.5 : Prove that the three angles of an equilateral triangle are equal.

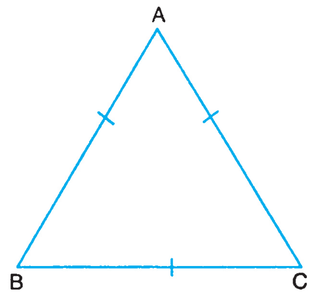


Fig. 11.27

Solution :

Given : An equilateral Δ ABC

To prove : $\angle A = \angle B = \angle C$

Proof : AB = AC (Given)

$\therefore$ $\angle C = \angle B$ (Angles opposite equal sides) ...(i)

Also AC = BC (Given)

$\therefore$ $\angle B = \angle A$ ...(ii)

From (i) and (ii),

$\angle A = \angle B = \angle C$

Hence the result.

Example 11.6 : ABC is an isosceles triangle in which AB = AC

(Fig. 11.28), If BD $\perp$ AC and CE $\perp$ AB, proe that BD = CE.

Solution : In ΔBDC and ΔCEB

$\angle BDC = \angle CEB$ (Measure of each is 90^o)

$\angle DCB = \angle EBC$ (Angles opposite equal sides of a triangle)

and $BC = CB$ (Common)

$\therefore$ $\Delta BDC \cong \Delta CEB$ (AAS)

Hence $BD = CE$ (c.p.c.t.)

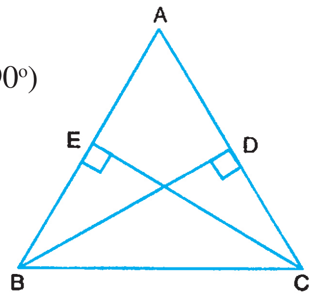


Fig. 11.28

This result can be stated in the following manner:

Perpendiculars (altitudes) drawn to equal sides, from opposite vertices of an isosceles triangle are equal.

The result can be extended to an equilateral triangle after which we can say that all the three altitudes of an equilateral triangle are equal.

Example : 11.7 : In Δ ABC (Fig. 11.29), D and E are mid-points of AC and AB respectively.

If AB = AC, then prove that BD = CE.

Solution : $BE = \frac{1}{2} AB$

and $CD = \frac{1}{2} AC$

$\therefore$ $BE = CD$ ...(i)

In Δ BEC and Δ CDB,

$BE = CD$ [By (i)]

$BC = CB$ (Common)

and $\angle EBC = \angle DCB$ ($\because$ AB = AC)

$\therefore$ $\Delta BEC \cong \Delta CDB$ (SAS)

Hence, CE = BD (c.p.c.t)

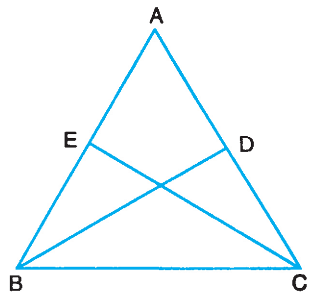


Fig. 11.29

Example 11.8 : In Δ ABC (Fig. 11.30) AB = AC and $\angle DAC = 124^o$; find the angles of the triangle.

Solution $\angle BAC = 180^o - 124^o = 56^o$

$\angle B = \angle C$

(Angles opposite to equal sides of a triangle)

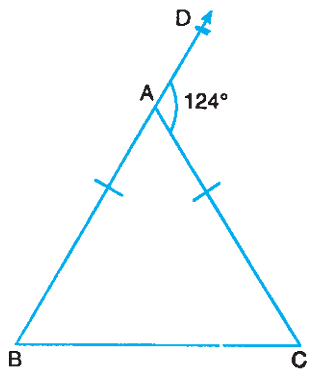


Fig. 11.30

Also $\angle B + \angle C = 124^{o}$

$$\angle B = \angle C = \frac{124^0}{2} = 62^0$$

Hence $\angle A = 56^{o}$, $\angle B = 62^{o}$, and $\angle C = 62^{o}$

CHECK YOUR PROGRESS 11.2

1. In Fig. 11.31, PQ = PR and SQ = SR. Prove that $\angle PQS = \angle PRS$.

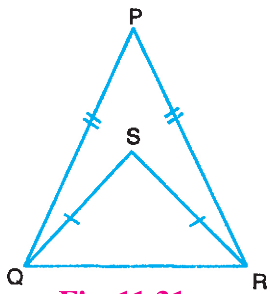


Fig. 11.31

2. Prove that ΔABC is an isosceles triangle, if the altitude AD bisects the base BC (Fig. 11.32).

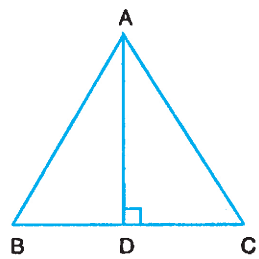


Fig. 11.32

3. If the line l in Fig. 11.33 is parallel to the base BC of the isosceles ΔABC, find the angles of the triangle.

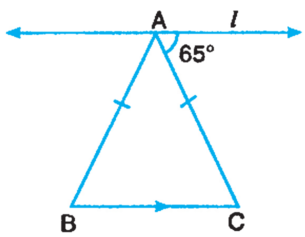


Fig. 11.33

4. ΔABC is an isosceles triangle such that AB = AC. Side BA is produced to a point D such that AB = AD. Prove that $\angle BCD$ is a right angle.

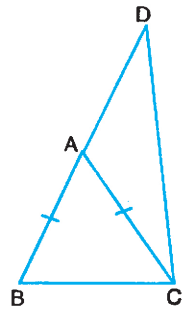


Fig. 11.34

5. In Fig. 11.35. D is the mid point of BC and perpendiculars DF and DE to sides AB and AC respectively are equal in length. Prove that ΔABC is an isosceles triangle.

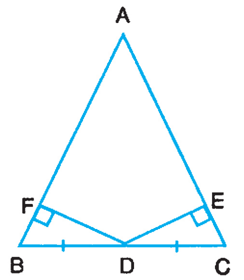


Fig. 11.35

6. In Fig. 11.36, PQ = PR, QS and RT are the angle bisectors of ∠Q and ∠R respectively. Prove that QS = RT.

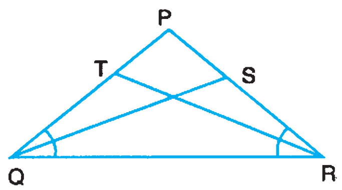


Fig. 11.36

7. ΔPQR and ΔSQR are isosceles triangles on the same base QR (Fig. 11.37). Prove that ∠PQS = ∠PRS.

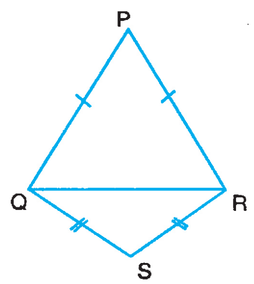


Fig. 11.37

8. In ΔABC, AB = AC (Fig. 11.38). P is a point in the interior of the triangle such that ∠ABP = ∠ACP . Prove that AP bisects ∠BAC.

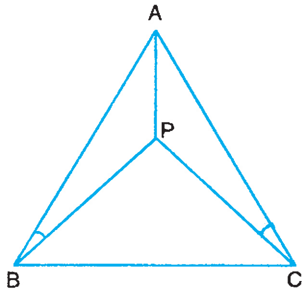


Fig. 11.38

11.5 INEQUALITIES IN A TRIANGLE

We have learnt the relationship between sides and angles of a triangle when they are equal. We shall now study some relations among sides and angles of a triangle, when they are unequal.

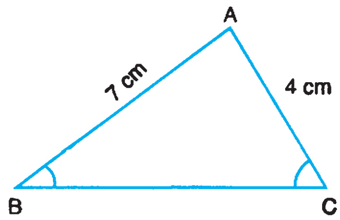


Fig. 11.39

In Fig. 11.39, triangle ABC has side AB longer than the side AC. Measure $\angle B$ and $\angle C$. You will find that these angles are not equal and $\angle C$ is greater than $\angle B$. If you repeat this experiment, you will always find that this observation is true. This can be proved easily, as follows.

11.5.1 Theorem

If two sides of a triangle are unequal, then the longer side has the greater angle opposite to it.

Given. A triangle ABC in which AB > AC.

To prove. $\angle ACB > \angle ABC$

Construction. Make a point D on the side AB such that AD = AC and join DC.

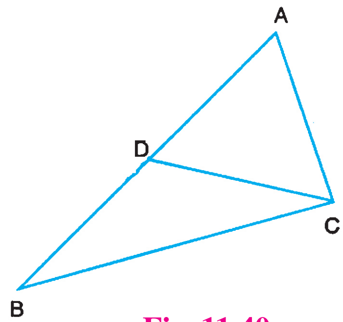


Fig. 11.40

Proof: In ΔACD,

AD = AC

$\therefore$ $\angle ACD = \angle ADC$ (Angles opposite equal sides)

But $\angle ADC > \angle ABC$

(Exterior angle of a triangle is greater than opposite interior angle)

Again $\angle ACB > \angle ACD$ (Point D lies in the interior of the $\angle ACB$).

$\therefore$ $\angle ACB > \angle ABC$

What can we say about the converse of this theorem. Let us examine.

In ΔABC, (Fig. 11.41) compare $\angle C$ and $\angle B$. It is clear that $\angle C$ is greater than $\angle B$. Now compare sides AB and AC opposite to these angles by measuring them. We observe that AB is longer than AC.

Again compare $\angle C$ and $\angle A$ and measure sides AB and BC opposite to these angles. We observe that $\angle C > \angle A$ and AB > BC; i.e. side opposite to greater angle is longer.

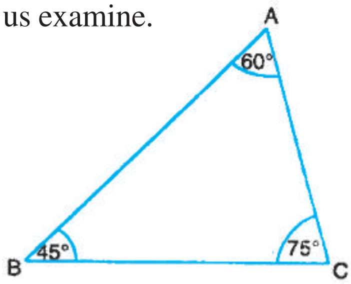


Fig. 11.41

Comparing $\angle A$ and $\angle B$, we observe a similar result. $\angle A > \angle B$ and BC > AC; i.e. side opposite to greater angle is longer.

You can also verify this property by drawing any type of triangle, a right triangle or an obtuse triangle.

Measure any pair of angles in a triangle. Compare them and then compare the sides opposite to them by measurement. You will find the above result always true, which we state as a property.

In a triangle, the greater angle has longer side opposite to it.

Observe that in a triangle if one angle is right or an obtuse then the side opposite to that angle is the longest.

You have already learnt the relationship among the three angles of a triangle i.e., the sum of the three angles of a triangle is 180°. We shall now study whether the three sides of a triangle are related in some way.

Draw a triangle ABC.

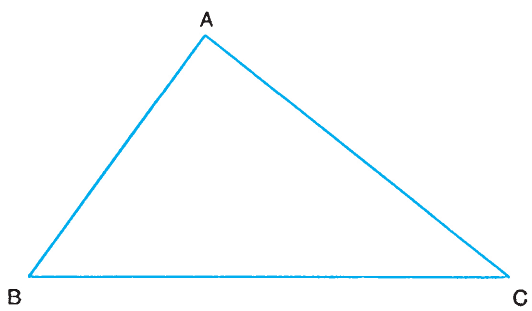


Fig. 11.42

Measure its three sides AB, BC and CA.

Now find the sum of different pairs AB+BC, BC+CA, and CA+AB separately and compare each sum of a pair with the third side, we observe that

(i) AB + BC>CA

(ii) BC + CA > AB and

(iii) CA + AB > BC

Thus we conclude that

Sum of any two sides of a triangle is greater than the third side.

ACTIVITY

Fix three nails P, Q and R on a wooden board or any surface.

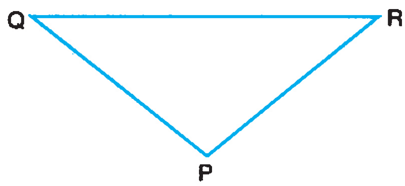


Fig. 11.43

Take a piece of thread equal in length to QR and another piece of thread equal in length (QP + PR). Compare the two lengths, you will find that the length corresponding to (QP + PR) > the length corresponding to QR confirming the above property.

Example 11.9 : In which of the following four cases, is construction of a triangle possible from the given measurements

(a) 5 cm, 8 cm and 3 cm

(b) 14 cm, 6 cm and 7 cm

(c) 3.5 cm, 2.5 cm and 5.2 cm

(d) 20 cm, 25 cm and 48 cm.

Solution. In (a) $5 + 3 \ngtr 8$, in (b) $6 + 7 \ngtr 14$

in (c) $3.5 + 2.5 > 5.2$, $3.5 + 5.2 > 2.5$ and $2.5 + 5.2 > 3.5$ and

in (d) $20 + 25 \ngtr 48$.

Ans. (c)

Example 11.10 : In Fig. 11.44, AD is a median of ΔABC. Prove that $AB + AC > 2AD$.

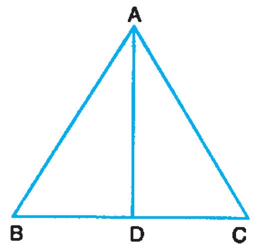


Fig. 11.44

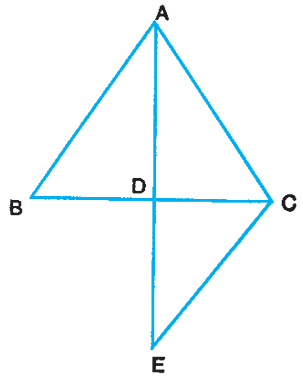


Fig. 11.45

Solution: Produce AD to E such that AD = DE and join C to E.

Consider ΔABD and ΔECD

Here, $BD = CD$

$\angle ADB = \angle EDC$

and $AD = ED$

$\therefore$ $\Delta ABC \cong \Delta ECD$

$\therefore$ $AB = EC$

Now in ΔACE,

$EC + AC > AE$

or $AB + AC > 2AD$ $\quad$ ($\because$ $AD = ED \Rightarrow AE = 2AD$)

CHECK YOUR PROGRESS 11.3

1. PQRS is a quadrilateral in which diagonals PR and QS intersect at O. Prove that $PQ + QR + RS + SP > PR + QS$.

2. In triangle ABC, AB = 5.7 cm, BC = 6.2 cm and CA = 4.8 cm. Name the greatest and the smallest angle.

3. In Fig. 11.46, if $\angle CBD > \angle BCE$ then prove that $AB > AC$.

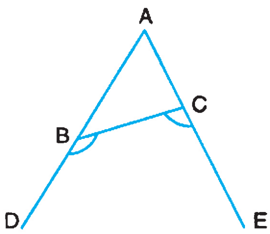


Fig. 11.46

4. In Fig. 11.47, D is any point on the base BC of a ΔABC. If AB > AC then prove that AB > AD.

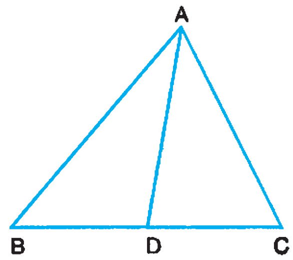


Fig. 11.47

5. Prove that the sum of the three sides of triangle is greater than the sum of its three medians.

   (Use Example 11.10)

6. In Fig. 11.48, if AB = AD then prove that BC > CD.

   [Hint : $\angle ADB = \angle ABD$].

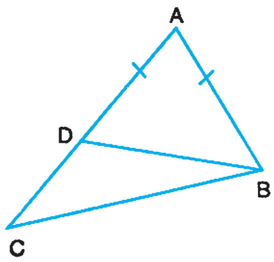


Fig. 11.48

7. In Fig. 11.49, AB is parallel to CD. If $\angle A > \angle B$ then prove that BC > AD.

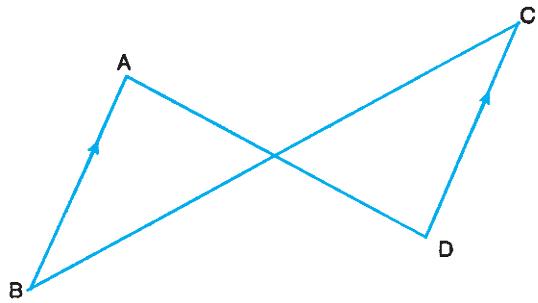


Fig. 11.49

LET US SUM UP

- Figures which have the same shape and same size are called congruent figures.
- Two congruent figures, when placed one over the other completely cover each other. All parts of one figure are equal to the corresponding parts of the other figure.

- To prove that two triangles are congruent we need to know the equality of only three corresponding parts. These corresponding parts must satisfy one of the four criteria.
  (i) SAS (ii) ASA or AAS
  (iii) SSS (iv) RHS
- Angles opposite to equal sides of a triangle are equal.
- Sides opposite to equal angles of a triangle are equal.
- If two sides of a triangle are unequal, then the longer side has the greater angle opposite to it.
- In a triangle, the greater angle has the longer side opposite to it.
- Sum of any two sides of a triangle is greater than the third side.

TERMINAL EXERCISE

1. Two lines AB and CD bisect each other at O. Prove that CA = BD (Fig. 11.50)

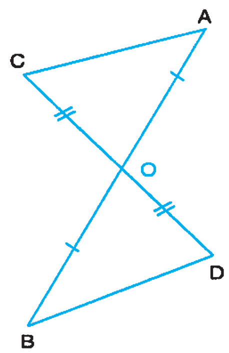


Fig. 11.50

2. In a ΔABC, if the median AD is perpendicular to the base BC then prove that the triangle is an isosceles triangle.

3. In Fig. 11.51, ΔABC and ΔCDE are such that BC = CE and AB = DE. If $\angle B = 60^o$, $\angle ACE = 30^o$ and $\angle D = 90^o$, then prove that the two triangles are congruent.

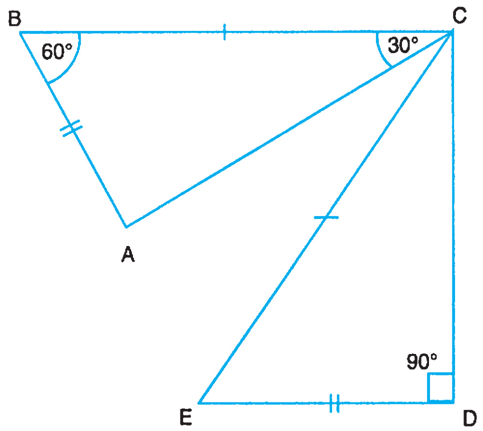


Fig. 11.51

4. In Fig. 11.52 two sides AB and BC and the altitude AD of ΔABC are respectively equal to the sides PQ and QR and the altitude PS, Prove that $\Delta ABC \cong \Delta PQR$.

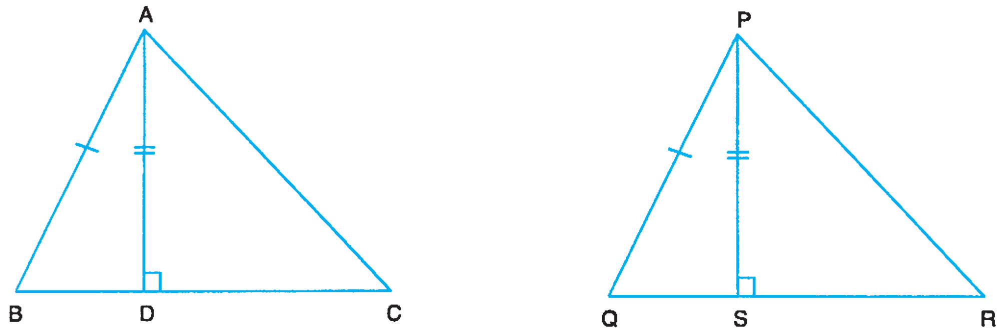


Fig. 11.52

5. In a right triangle, one of the acute angles is 30^o. Prove that the hypotenuse is twice the side opposite to the angle of 30^o.

6. Line segments AB and CD intersect each other at O such that O is the midpoint of AB. If AC is parallel to DB then prove that O is also the mid piont of CD.

7. In Fig. 11.53, AB is the longest side and DC is the shortest side of a quadrilateral ABCD. Prove that $\angle C > \angle A$ and $\angle D > \angle B$. [Hint : Join AC and BD].

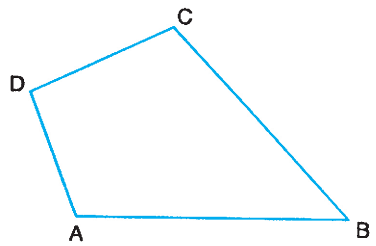


Fig. 11.53

8. ABC is an isosceles triangle in which AB = AC and AD is the altitude from A to the base BC. Prove that BD = DC.

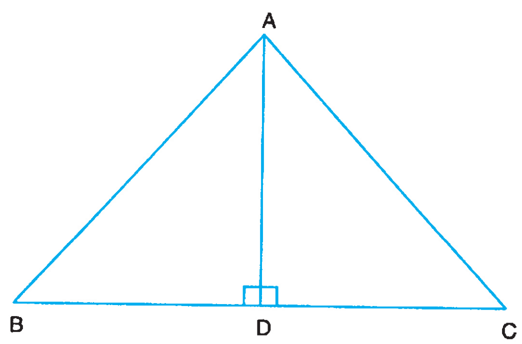


Fig. 11.54

9. Prove that the medians bisecting the equal sides of an isosceles triangle are also equal. [Hint : Show that $\Delta DBC \cong \Delta ECB$]

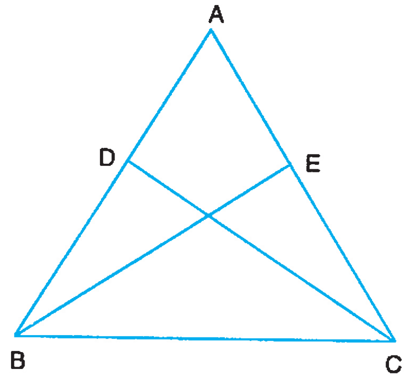


Fig. 11.55

ANSWERS TO CHECK YOUR PROGRESS

11.1

1. (a)	2. (b)
3. (b)	4. (c)

8. $\angle P = \angle C$ $\angle Q = \angle A$ and $\angle R = \angle B$.

11.2

3. $\angle B = \angle C = 65^o$, $\angle A = 50^o$

11.3

2. Greatest angle is A and smallest angle is B.

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12

CONCURRENT LINES

You have already learnt about concurrent lines, in the lesson on lines and angles. You have also studied about triangles and some special lines, i.e., medians, right bisectors of sides, angle bisectors and altitudes, which can be drawn in a triangle. In this lesson, we shall study the concurrency property of these lines, which are quite useful.

OBJECTIVES

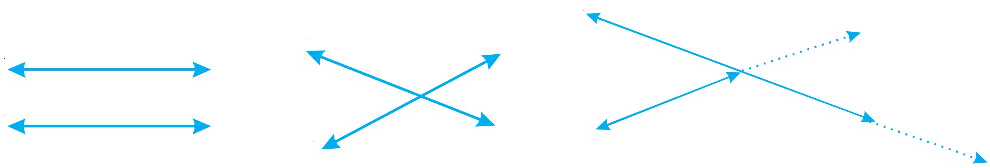
After studying this lesson, you will be able to

- *define the terms concurrent lines, median, altitude, angle bisector and perpendicular bisector of a side of a triangle.*
- *Verify the concurrence of medians, altitudes, perpendicular bisectors of sides and angle bisectors of a triangle.*

EXPECTED BACKGROUND KNOWLEDGE

Properties of intersecting lines, such as:

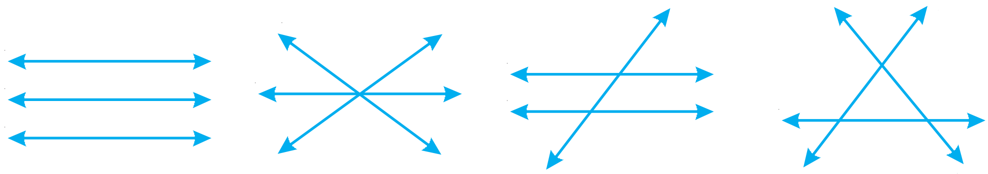
- Two lines in a plane can either be parallel [See Fig 12.1 (a)] or intersecting (See Fig. 12.1 (b) and (c)].



(a) (b) (c)

Fig. 12.1

- Three lines in a plane may
  - (i) be paralled to each other, i.e., intersect in no point [See Fig. 12.2 (a)] or
  - (ii) intersect each other in exactly one point [Fig. 12.2(b)], or
  - (iii) intersect each other in two points [Fig. 12.2(c)], or
  - (iv) intersect each other at the most in three points [Fig. 12.2(d)]

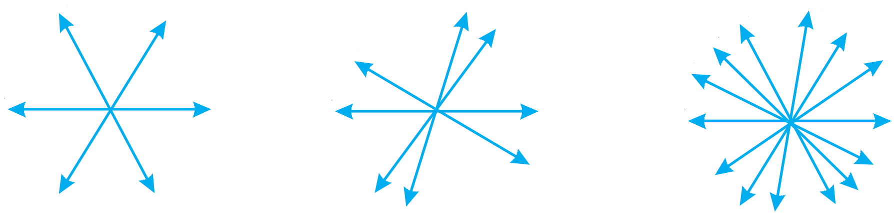


(a) (b) (c) (d)

Fig. 12.2

12.1 CONCURRENT LINES

Three lines in a plane which intersect each other in exactly one point or which pass through the same point are called **concurrent lines** and the common **point** is called the **point of concurrency** (See Fig. 12.3).



(a) (b) (c)

Fig. 12.3

12.1.1 Angle Bisectors of a Triangle

In triangle ABC, the line AD bisects ∠A of the triangle. (See Fig. 12.4)

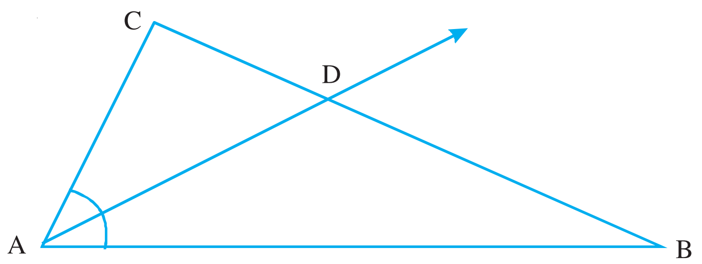
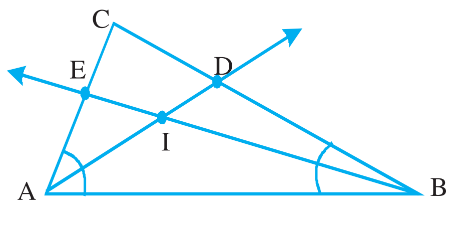


Fig. 12.4

A line which bisects an angle of a triangle is called an **angle bisector** of the triangle.

How many angle bisectors can a triangle have? Since a triangle has three angles, we can draw three angle bisectors in it. AD is one of the three angle bisectors of ΔABC. Let us draw second angle bisector BE of $\angle B$ (See Fig. 12.5)



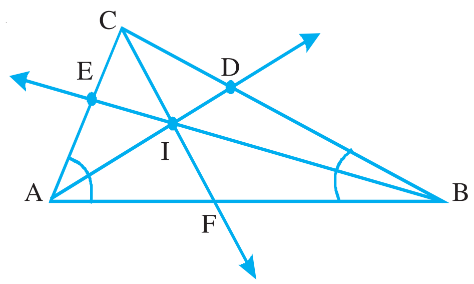


Fig. 12.5

Fig. 12.6

The two angle bisectors of the ΔABC intersect each other at I. Let us draw the third angle bisector CF of $\angle C$ (See Fig. 12.6). We observe that this angle bisector of the triangle also passes through I. In other words they are concurrent and the point of concurrency is I.

We may take any type of triangle—acute, right or obtuse triangle, and draw its angle bisectors, we will always find that the three angle bisectors of a triangle are concurrent (See Fig. 12.7)

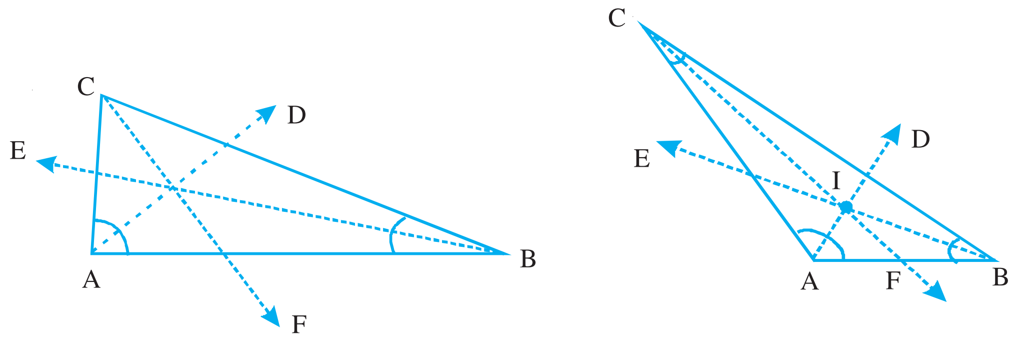


Fig. 12.7

Thus we conclude the following:

Angle bisectors of a triangle pass through the same point, that is they are concurrent

The point of concurrency I is called the 'Incentre' of the triangle.

Can you reason out, why the name incentre for this point?

Recall that the locus of a point equidistant from two intersecting lines is the pair of angle bisectors of the angles formed by the lines. Since I is a point on the bisector of $\angle BAC$, it must be equidistant from AB and AC. Also I is a point on angle bisector of $\angle ABC$, (See

Fig. 12.8), it must also be equidistant from AB and BC. Thus point of concurrency I is at the same distance from all the three sides of the triangle.

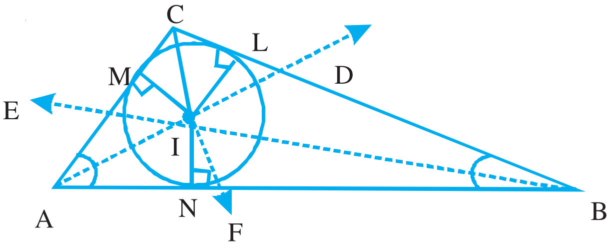


Fig. 12.8

Thus, we have IL = IM = IN (Fig. 12.8). Taking I as the centre and IL as the radius, we can draw a circle touching all the three sides of the triangle called **'Incircle'** of the triangle. I being the centre of the incircle is called the **Incentre** and IL, the radius of the incircle is called the inradius of the triangle.

Note: The incentre always lies in the interior of the triangle.

12.1.2: Perpendicular Bisectors of the Sides of a Triangle

ABC is a triangle, line DP bisects side BC at right angle. A line which bisects a side of a triangle at right angle is called the perpendicular bisector of the side. Since a triangle has three sides, so we can draw three perpendicular bisectors in a triangle. DP is one of the three perpendicular bisectors of ΔABC (Fig. 12.9). We draw the second perpendicular bisector EQ, intersecting DP at O (Fig. 12.10). Now if we also draw the third perpendicular bisector FR, then we observe that it also passes through the point O (Fig. 12.11). In other words, we can say that the three perpendicular bisectors of the sides are concurrent at O.

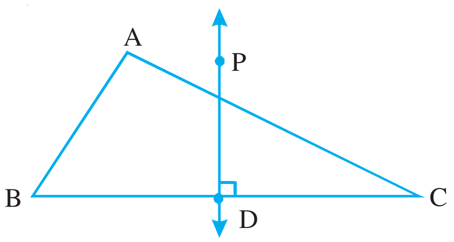


Fig. 12.9

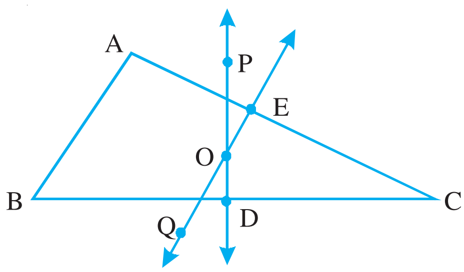


Fig. 12.10

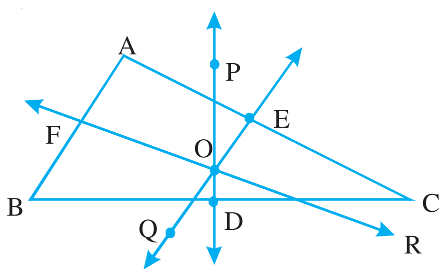
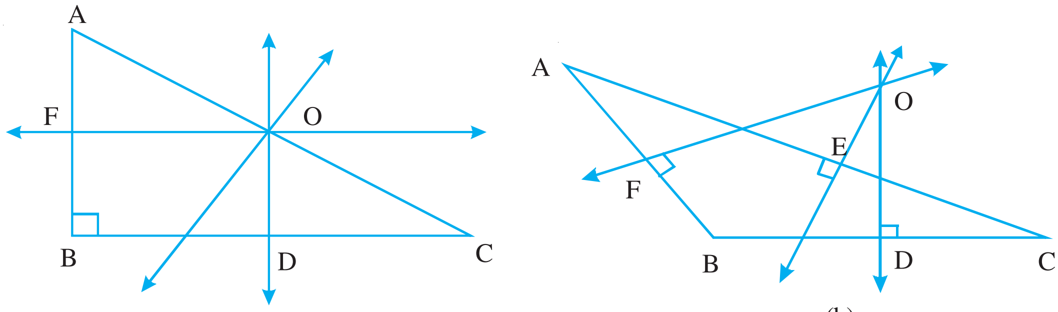


Fig. 12.11

We may repeat this experiment with any type of triangle, but we will always find that the three perpendicular bisectors of the sides of a triangle pass through the same point.



(a) (b)

Fig. 12.12

Thus we conclude that:

The three perpendicular bisectors of the sides of a triangle pass through the same point, that is, they are concurrent.

The point of concurrency O is called the 'circumcentre' of the triangle

Can you reason out: why the name circumcentre for this point?

Recall that the locus of a point equidistant from two given points is the perpendicular bisector of the line joining the two points. Since O lies on the perpendicular bisector of BC, so it must be equidistant from both the point B and C i.e., BO = CO (Fig. 12.13).

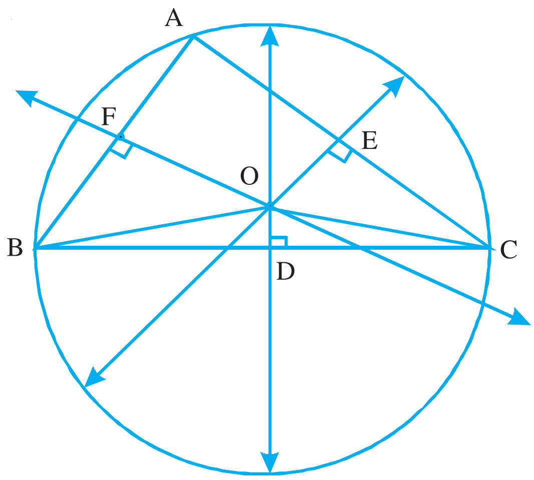


Fig. 12.13

The point O also lies on the perpendicular bisector of AC, so it must be equidistant from both A and C, that is, AO = CO. Thus, we have AO = BO = CO.

If we take O as the centre and AO as the radius, we can draw a circle passing through the three vertices, A, B and C of the triangle, called '**Circumcircle**' of the triangle. O being the centre of this circle is called the **circumcentre** and AO the radius of the circumcircle is called **circumradius** of the triangle.

Note that the circumcentre will be

1. in the interior of the triangle for an acute triangle (Fig. 12.11)
2. on the hypotenuse for a right triangle [Fig. 12.12(a)]
3. in the exterior of the triangle for an obtuse triangle [Fig. 12.12(b)].

12.1.3 Altitudes of a Triangle

In ΔABC, the line AL is the perpendicular drawn from vertex A to the opposite side BC. (Fig. 12.14).

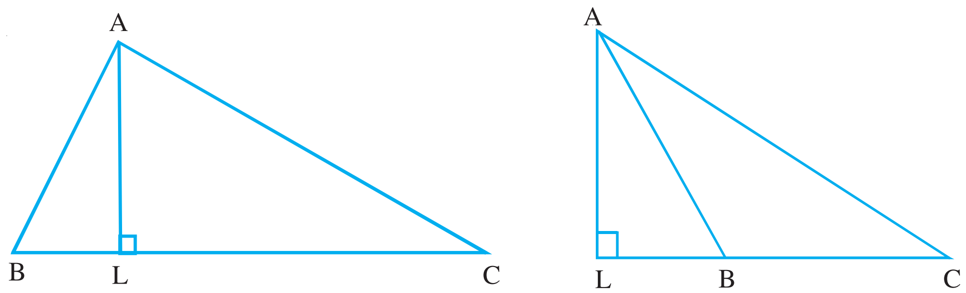


Fig. 12.14

Perpendicular drawn from a vertex of a triangle to the oposite side is called its altitude. How many altitudes can be drawn in a triangle? There are three vertices in a triangle, so we can draw three of its altitudes. AL is one of these altitudes. Now we draw the second altitude BM, which intersects the first altitude at a point H (see Fig. 12.15). We also draw the third altitude CN and observe that it also passes through the point H (Fig. 12.16). This shows that the three altitudes of the triangle pass through the same point.

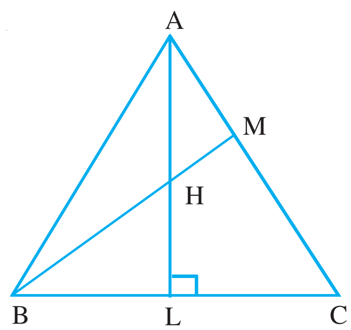


Fig. 12.15

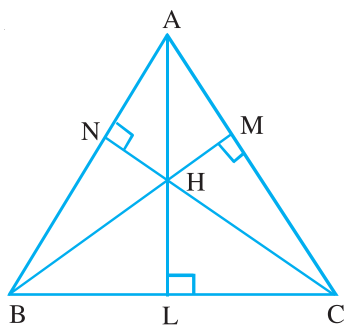


Fig. 12.16

We may take any type of triangle and draw its three altitudes. We always find that the three altitudes of a triangle are concurrent.

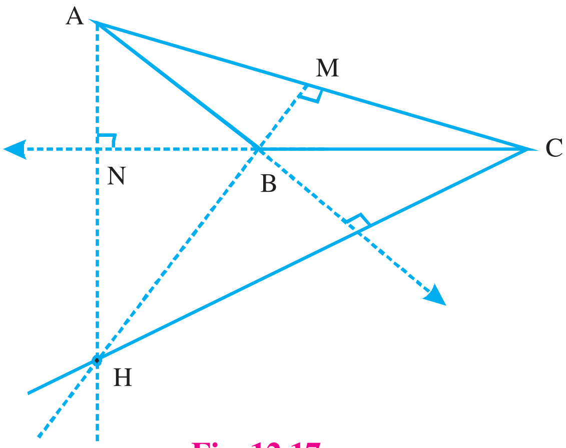


Fig. 12.17

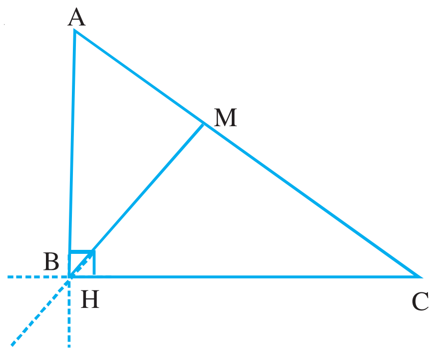


Fig. 12.18

Thus we conclude that:

In a triangle, the three altitudes pass through the same point, that is, they are concurrent.

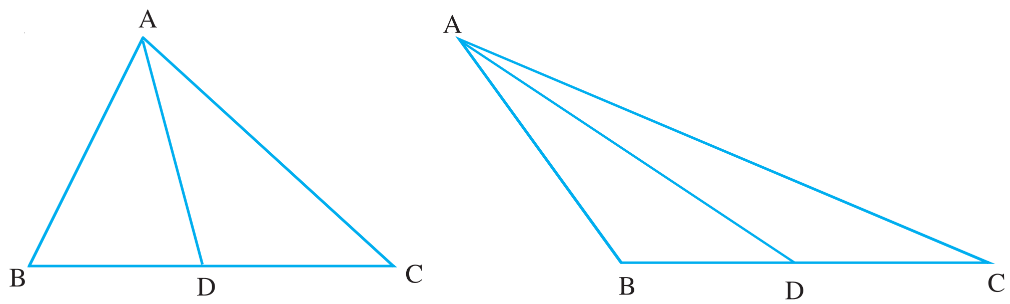
The point of concurrency is called the 'Orthocentre' of the triangle.

Again observe that the orthocentre will be

1. in the interior of the triangle for an acute triangle (Fig. 12.16)
2. in the exterior of the triangle for an obtuse triangle (Fig. 12.17)
3. at the vertex containing the right angle for a right triangle (Fig. 12.18)

12.1.4 Medians of a Triangle

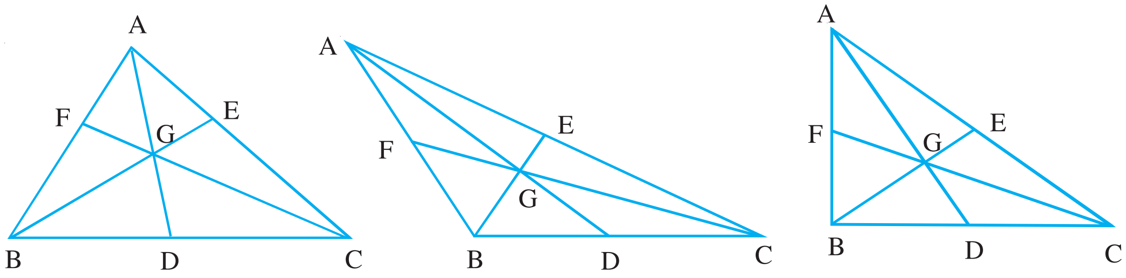
In ΔABC, AD joins the vertex A to the mid point D of the opposite side BC (Fig. 12.19)



(a) (b)

Fig. 12.19

A line joining a vertex to the mid point of the opposite side of a triangle is called its median. Clearly, three medians can be drawn in a triangle. AD is one of the medians. If we draw all the three medians in any triangle, we always find that the three medians pass through the same point [Fig. 12.20 (a), (b), (c)]



(a) (b) (c)

Fig. 12.20

Here in each of the triangles ABC given above (Fig. 12.20) the three medians AD, BE and CF are concurrent at G.. In each triangle we measure the parts into which G divides each median. On measurement, we observe that

AG = 2GD, BG = 2GE

and CG = 2 GF

that is, the point of concurrency G divides each of the medians in the ratio 2 : 1.

Thus we conclude that:

Medians of a triangle pass through the same point, which divides each of the medians in the ratio 2 : 1.

The point of concurrency G is called the 'centroid' of the triangle.

ACTIVITY FOR YOU

Cut out a triangle from a piece of cardboard. Draw its three medians and mark the centroid G of the triangle. Try to balance the triangle by placing the tip of a pointed stick or a needle of compasses below the point G or at G. If the position of G is correctly marked then the weight of the triangle will balance at G (Fig. 12.21).

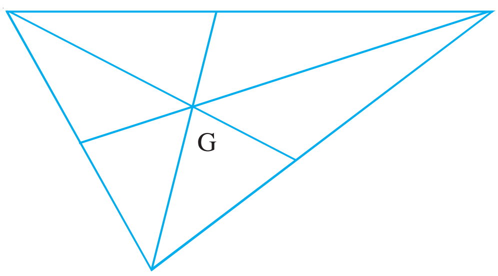


Fig. 12.21

Can you reason out, why the point of concurrency of the medians of a triangle is called its centroid. It is the point where the weight of the triangle is centered or it is the point through which the weight of the triangle acts.

We consider some examples using these concepts.

Example 12.1: In an isosceles triangle, show that the bisector of the angle formed by the equal sides is also a perpendicular bisector, an altitude and a median of the triangle.

Solution: In ΔABD and ΔACD,

$AB = AC$ (Given)

$\angle BAD = \angle CAD$ [$\because$ AD is bisector of $\angle A$]

and $AD = AD$

$\therefore$ $\Delta ABD \cong \Delta ACD$

$\therefore$ $BD = CD$

$\Rightarrow$ AD is also a median

$\Rightarrow$ Also $\angle ADB = \angle ADC = 90^\circ$

$\Rightarrow$ AD is an altitude

Since, $BD = DC$,

AD is perpendicular bisector of side BC.

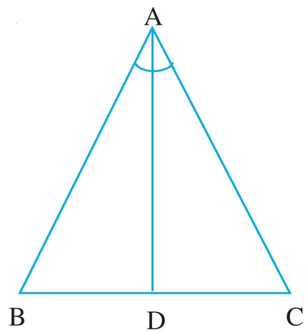


Fig. 12.22

Example 12.2: In an equilateral triangle, show that the three angle bisectors are also the three perpendicular bisectors of sides, three altitudes and the three medians of the triangle.

Solution: Since $AB = AC$

Therefore, AD, the bisector of $\angle A$ is also a perpendicular bisector of BC, an altitude and a median of the ΔABD.

(Refer Example 12.1 above)

Similarly, since $AB = BC$ and $BC = AC$

$\therefore$ BE and CF, angle bisectors of $\angle B$ and $\angle C$ respectively, are also perpendicular bisectors, altitudes and medians of the ΔABC.

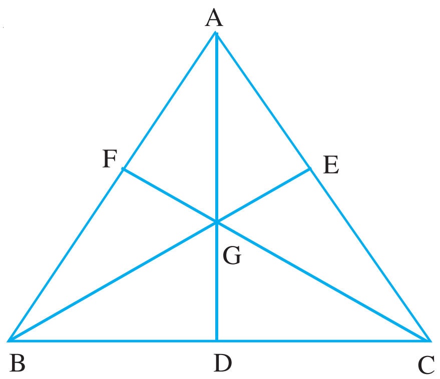


Fig. 12.23

Example 12.3: Find the circumradius of circumcircle and inradius of incircle of an equilateral triangle of side *a*.

Solution: We draw perpendicular from the vertex A to the side BC.

AD is also the angle bisector of $\angle A$, perpendicular bisector of side BC and a median joining vertex to the midpoint of BC.

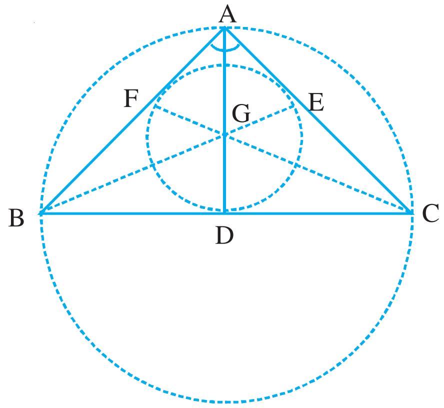


Fig. 12.24

$\therefore \quad AD = \dfrac{\sqrt{3}}{2}a, \text{ as } BC = a$

$\Rightarrow \quad AG = \text{circumradius in this case} = \dfrac{2}{3} \times \dfrac{\sqrt{3}}{2}a = \dfrac{\sqrt{3}}{3}a$

and $\quad GD = \text{inradius in this case} = \dfrac{1}{3} \times \dfrac{\sqrt{3}}{2}a = \dfrac{\sqrt{3}}{6}a$.

CHECK YOUR PROGRESS 12.1

1. In the given figure BF = FC, $\angle BAE = \angle CAE$ and $\angle ADE = \angle GFC = 90^{o}$, then name a median, an angle bisector, an altitude and a perpendicular bisector of the triangle.

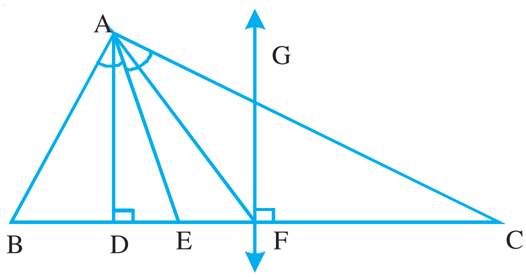


Fig. 12.25

2. In an equilateral triangle show that the incentre, the circumcentre, the orthocentre and the centroid are the same point.

3. In an equilateral ΔABC (Fig. 12.26). G is the centroid of the triangle. If AG is 4.8 cm, find AD and BE.

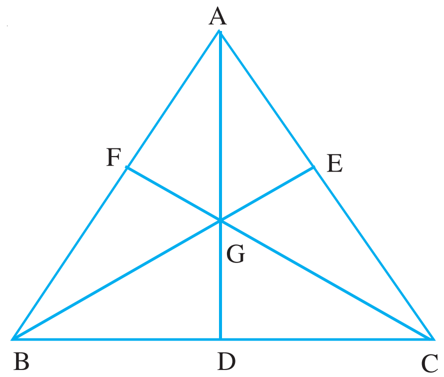


Fig. 12.26

4. If H is the orthocentre of ΔABC, then show that A is the orthocentre of the ΔHBC.
5. Choose the correct answers out of the given alternatives in the following questions:
   (i) In a plane, the point equidistant from vertices of a triangle is called its
   (a) centroid (b) incentre
   (c) circumcentre (d) orthocentre
   (ii) In the plane of a triangle, the point equidistant from the sides of the triangle is called its
   (a) centroid (b) incentre
   (c) circumcentre (d) orthocentre

LET US SUM UP

- Three or more lines in a plane which intersect each other in exactly one point are called concurrent lines.
- A line which bisects an angle of a triangle is called an angle bisector of the triangle.
- A line which bisects a side of a triangle at right angle is called the perpendicular bisector of the side of the triangle.
- A line drawn perpendicular from a vertex of a triangle to its opposite side is called an altitude of the triangle.
- A line which joins a vertex of a triangle to the mid-point of the opposite side is called a median of the triangle.
- In a triangle
  (i) angle bisectors are concurrent and the point of concurrency is called **incentre.**

(ii) perpendicular bisectors of the sides are concurrent and the point of concurrency is called **circumcentre.**

(iii) altitudes are concurrent and the point of concurrency is called **orthocentre.**

(iv) medians are concurrent and the point of concurrency is called **centroid,** which divides each of the medians in the ratio 2 : 1.

TERMINAL EXERCISE

1. In the given Fig. 12.27, D, E and F are the mid points of the sides of ΔABC. Show that $BE + CF > \frac{3}{2} BC$.

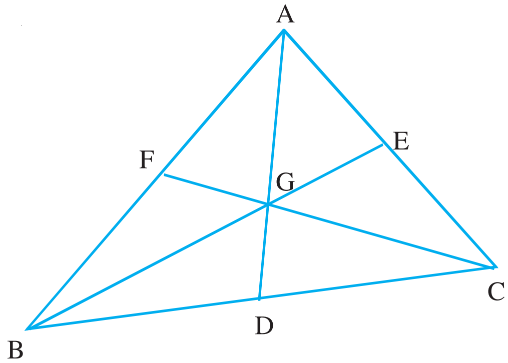


Fig. 12.27

2. ABC is an isoceles triangle such that AB = AC and D is the midpoint of BC. Show that the centroid, the incentre, the circumcentre and the orthocentre, all lie on AD.

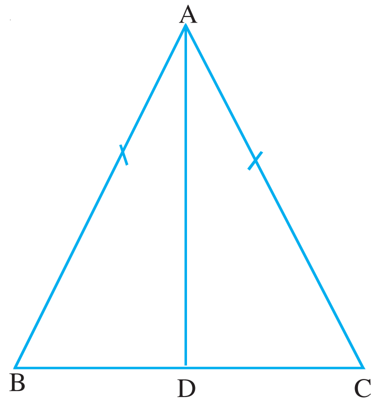


Fig. 12.28

3. ABC is an isoceles triangle such that AB = AC = 17 cm and base BC = 16 cm. If G is the centroid of ΔABC, find AG.

4. ABC is an equilateral triangle of side 12 cm. If G be its centroid, find AG.

ACTIVITIES FOR YOU

1. Draw a triangle ABC and find its circumcentre. Also draw the circumcircle of the triangle.
2. Draw an equilateral triangle. Find its incentre and circumcentre. Draw its incircle and circumcircle.
3. Draw the circumcircle and the incircle for an equilateral triangle of side 5 cm.

ANSWERS TO CHECK YOUR PROGRESS

12.1

1. Median - AF, Angle bisector AE

   Altitude - AD and perpendicular bisector - GF

3. AD = 7.2 cm, BE = 7.2 cm

5. (i) (c) (ii) (b)

ANSWERS TO TERMINAL EXERCISE

3. AG = 10 cm

4. AG = $4\sqrt{3}$ cm

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QUADRILATERALS

If you look around, you will find many objects bounded by four line-segments. Any surface of a book, window door, some parts of window-grill, slice of bread, the floor of your room are all examples of a closed figure bounded by four line-segments. Such a figure is called a quadrilateral.

The word quadrilateral has its origin from the two words "quadric" meaning four and "lateral" meaning sides. Thus, a quadrilateral is that geometrical figure which has four sides, enclosing a part of the plane.

In this lesson, we shall study about terms and concepts related to quadrilateral with their properties.

OBJECTIVES

After studying this lesson, you will be able to

- *describe various types of quadrilaterals viz. trapeziums, parallelograms, rectangles, rhombuses and squares;*
- *verify properties of different types of quadrilaterals;*
- *verify that in a triangle the line segment joining the mid-points of any two sides is parallel to the third side and is half of it;*
- *verify that the line drawn through the mid-point of a side of a triangle parallel to another side bisects the third side;*
- *verify that if there are three or more parallel lines and the intercepts made by them on a transversal are equal, the corresponding intercepts on any other transversal are also equal;*
- *verify that a diagonal of a parallelogram divides it into two triangles of equal area;*
- *solve problem based on starred results and direct numerical problems based on unstarred results given in the curriculum;*

- *prove that parallelograms on the same or equal bases and between the same parallels are equal in area;*
- *verify that triangles on the same or equal bases and between the same parallels are equal in area and its converse.*

EXPECTED BACKGROUND KNOWLEDGE

- Drawing line-segments and angles of given measure.
- Drawing circles/arcs of given radius.
- Drawing parallel and perpendicular lines.
- Four fundamental operations on numbers.

13.1 QUADRILATERAL

Recall that if A, B, C and D are four points in a plane such that no three of them are collinear and the line segments AB, BC, CD and DA do not intersect except at their end points, then the closed figure made up of these four line segments is called a quadrilateral with vertices A, B, C and D. A quadrilateral with vertices A, B, C and D is generally denoted by quad. ABCD. In Fig. 13.1 (i) and (ii), both the quadrilaterals can be named as quad. ABCD or simply ABCD.

In quadrilateral ABCD,

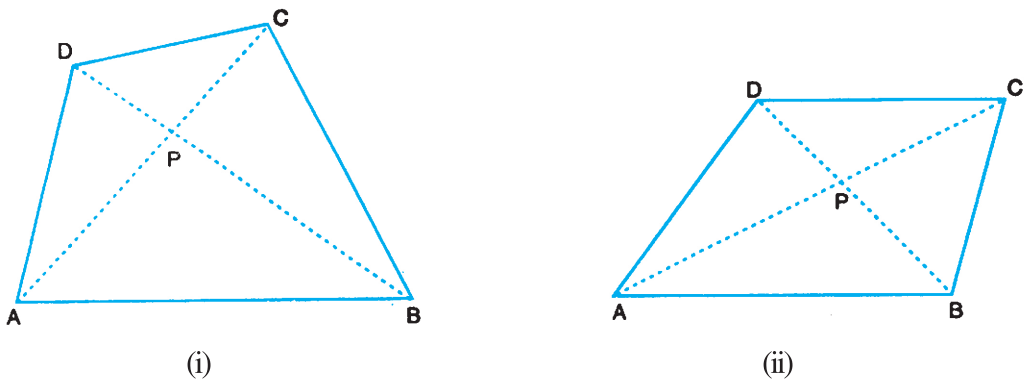

Fig. 13.1

(i) AB and DC ; BC and AD are two pairs of opposite sides.

(ii) $\angle$A and $\angle$C ; $\angle$B and $\angle$D are two pairs of opposite angles.

(iii) AB and BC ; BC and CD are two pairs of consecutive or adjacent sides. Can you name the other pairs of consecutive sides?

(iv) $\angle$A and $\angle$B ; $\angle$B and $\angle$C are two pairs of consecutive or adjacent angles. Can you name the other pairs of consecutive angles?

(v) AC and BD are the two diagonals.

In Fig. 13.2, angles denoted by 1, 2, 3 and 4 are the interior angles or the angles of the quad. ABCD. Angles denoted by 5, 6, 7 and 8 are the exterior angles of the quad. ABCD.

Measure $\angle 1$, $\angle 2$, $\angle 3$ and $\angle 4$.

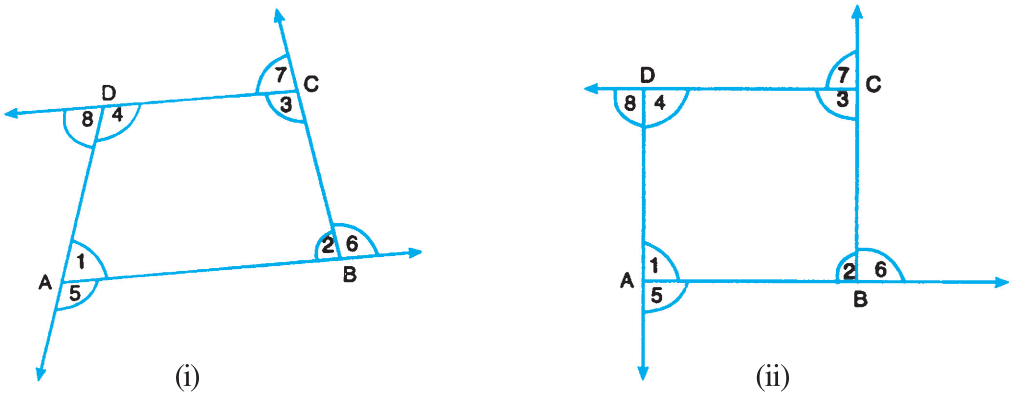

Fig. 13.2

What *is* the sum of these angles You will find that $\angle 1 + \angle 2 + \angle 3 + \angle 4 = 360°$.

i.e. sum of interior angles of a quadrilateral equals 360°.

Also what is the sum of exterior angles of the quadrilateral ABCD?

You will again find that $\angle 5 + \angle 6 + \angle 7 + \angle 8 = 360°$

i.e., sum of exterior angles of a quadrilateral is also 360°.

13.2 TYPES OF QUADRILATERALS

You are familiar with quadrilaterals and their different shapes. You also know how to name them. However, we will now study different types of quadrilaterals in a systematic way. A family tree of quadrilaterals is given in Fig. 13.3 below:

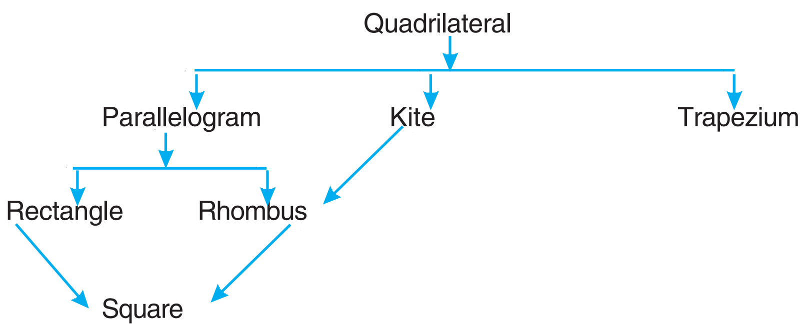

Fig. 13.3

Let us describe them one by one.

1. Trapezium

A quadrilateral which has only one pair of opposite sides parallel is called a trapezium. In

Fig. 13.4 [(i) and (ii)] ABCD and PQRS are trapeziums with AB || DC and PQ || SR respectively.

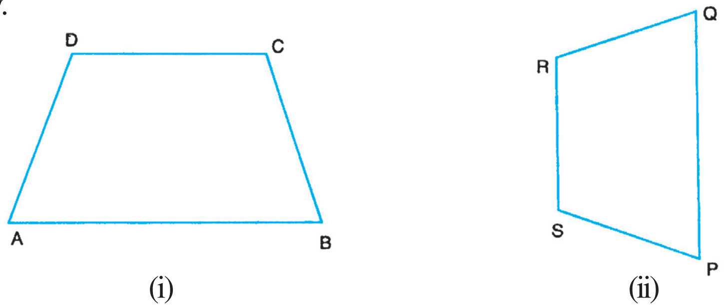

Fig. 13.4

2. Kite

A quadrilateral, which has two pairs of equal sides next to each other, is called a kite. Fig. 13.5 [(i) and (ii)] ABCD and PQRS are kites with adjacent sides AB and AD, BC and CD in (i) PQ and PS, QR and RS in (ii) being equal.

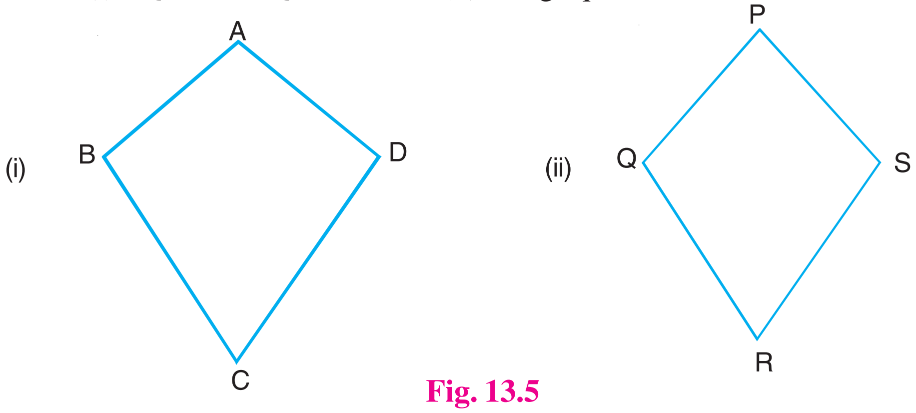

Fig. 13.5

3. Parallelogram

A quadrilateral which has both pairs of opposite sides parallel, is called a parallelogram. In Fig. 13.6 [(i) and (ii)] ABCD and PQRS are parallelograms with AB||DC, AD||BC and PQ||SR, SP||RQ. These are denoted by ||^{gm} ABCD (Parallelogram ABCD) and ||^{gm} PQRS (Parallelogram PQRS).

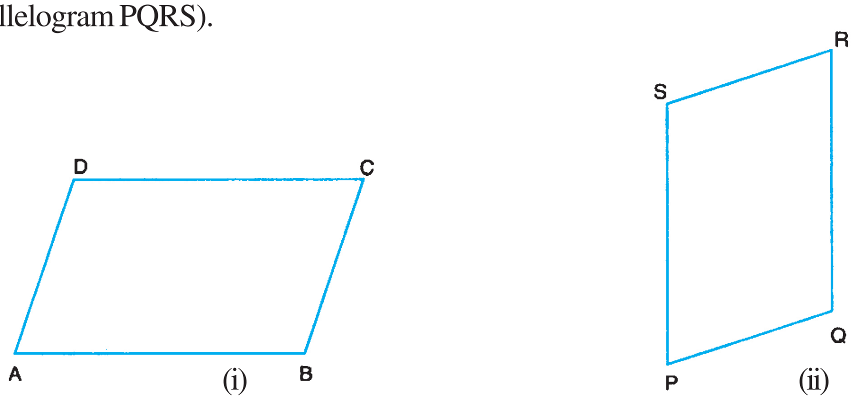

Fig. 13.6

4. Rhombus

A rhombus is a parallelogram in which any pair of adjacent sides is equal.

In Fig. 13.7 ABCD is a rhombus.

You may note that ABCD is a parallelogram with AB = BC = CD = DA i.e., each pair of adjacent sides being equal.

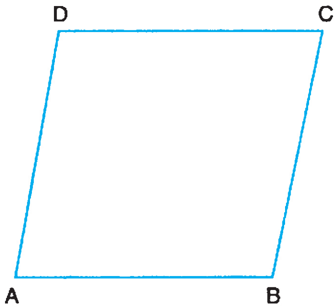

Fig. 13.7

5. Rectangle

A parallelogram one of whose angles is a right angle is called a rectangle.

In Fig. 13.8, ABCD is a rectangle in which AB||DC, AD||BC

and $\angle A = \angle B = \angle C = \angle D = 90^o$.

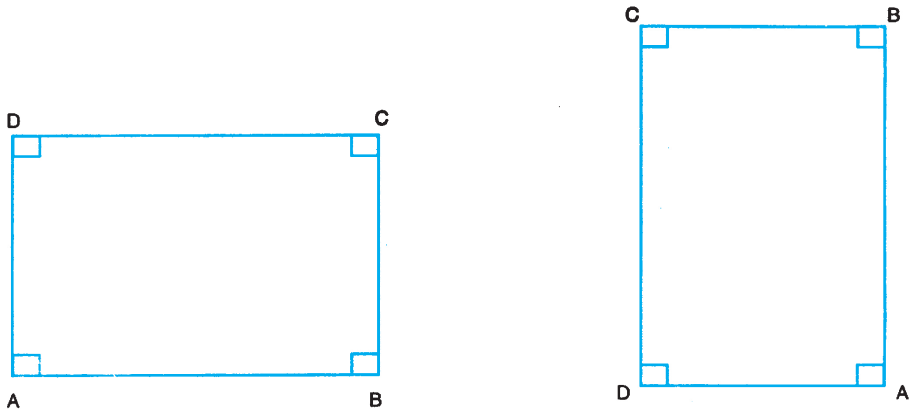

Fig. 13.8

6. Square

A square is a rectangle, with a pair of adjacent sides equal.

In other words, a parallelogram having all sides equal and each angle a right angle is called a square.

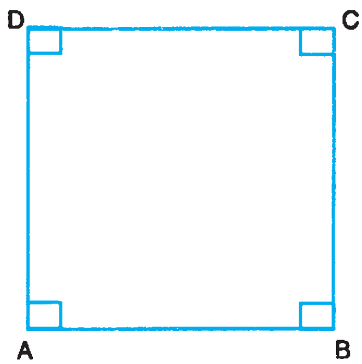

Fig. 13.9

In Fig. 13.9, ABCD is a square in which AB||DC, AD||BC, and AB = BC = CD = DA and $\angle A = \angle B = \angle C = \angle D = 90^\circ$.

Let us take some examples to illustrate different types of quadrilaterals.

Example 13.1: In Fig 13.10, PQR is a triangle. S and T are two points on the sides PQ and PR respectively such that ST||QR. Name the type of quadrilateral STRQ so formed.

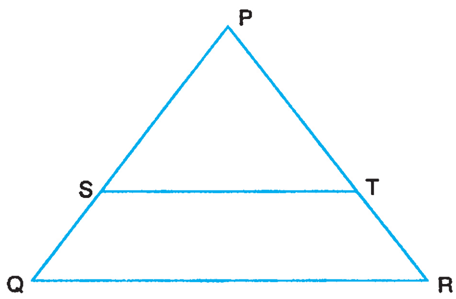

Fig. 13.10

Solution: Quadrilateral STRQ is a trapezium, because ST||QR.

Example 13.2: The three angles of a quadrilateral are 100°, 50° and 70°. Find the measure of the fourth angle.

Solution: We know that the sum of the angles of a quadrilateral is 360°.

Then $\quad 100^\circ + 50^\circ + 70^\circ + x^\circ = 360^\circ$

$$220^\circ + x^\circ = 360^\circ$$

$$x = 140$$

Hence, the measure of fourth angle is 140°.

CHECK YOUR PROGRESS 13.1

1. Name each of the following quadrilaterals.

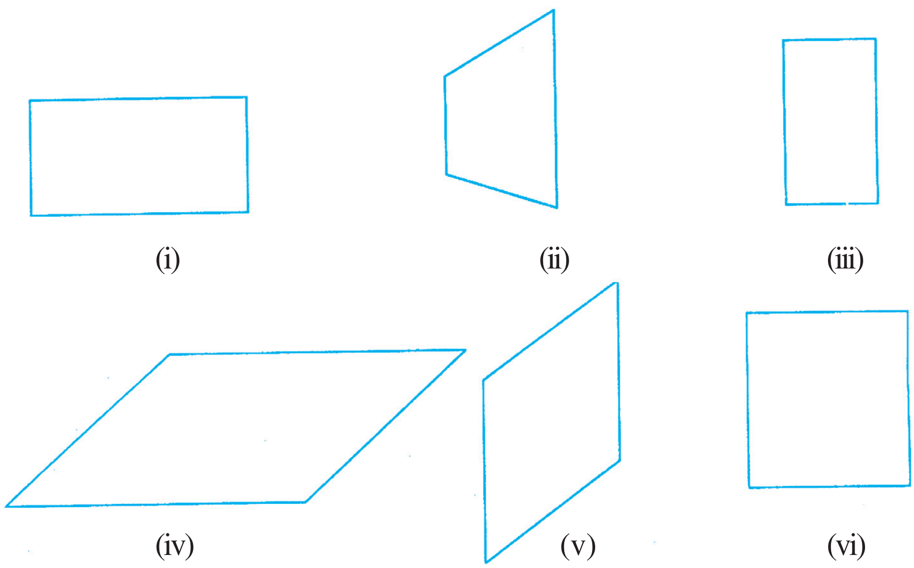

Fig. 13.10

2. State which of the following statements are correct ?

   (i) Sum of interior angles of a quadrilateral is 360°.

   (ii) All rectangles are squares,

   (iii) A rectangle is a parallelogram.

   (iv) A square is a rhombus.

   (v) A rhombus is a parallelogram.

   (vi) A square is a parallelogram.

   (vii) A parallelogram is a rhombus.

   (viii) A trapezium is a parallelogram.

   (ix) A trapezium is a rectangle.

   (x) A parallelogram is a trapezium.

3. In a quadrilateral, all its angles are equal. Find the measure of each angle.

4. The angles of a quadrilateral are in the ratio 5:7:7: 11. Find the measure of each angle.

5. If a pair of opposite angles of a quadrilateral are supplementary, what can you say about the other pair of angles?

13.3 PROPERTIES OF DIFFERENT TYPES OF QUADRILATERALS

1. Properties of a Parallelogram

We have learnt that a parallelogram is a quadrilateral with both pairs of opposite sides parallel. Now let us establish some relationship between sides, angles and diagonals of a parallelogram.

Draw a pair of parallel lines *l* and m as shown in Fig. 13.12. Draw another pair of parallel lines p and q such that they intersect *l* and m. You observe that a parallelogram ABCD is formed. Join AC and BD. They intersect each other at O.

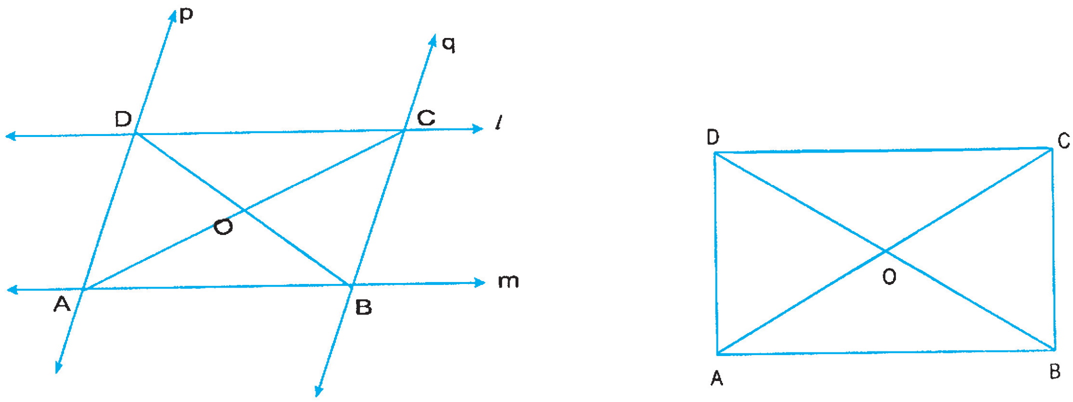

Fig. 13.12

Now measure the sides AB, BC, CD and DA. What do you find?

You will find that AB = DC and BC = AD.

Also measure ∠ABC, ∠BCD, ∠CDA and ∠DAB.

What do you find?

You will find that ∠DAB = ∠BCD and ∠ABC = ∠CDA

Again, Measure OA, OC, OB and OD.

What do you find?

You will find that OA = OC and OB = OD

Draw another parallelogram and repeat the activity. You will find that

The opposite sides of a parallelogram are equal.

The opposite angles of a parallelogram are equal.

The diagonals of a parallelogram bisect each other.

The above mentioned properties of a parallelogram can also be verified by Cardboard model which is as follows:

Let us take a cardboard. Draw any parallelogram ABCD on it. Draw its diagonal AC as shown in Fig 13.13 Cut the parallelogram ABCD from the cardboard. Now cut this parallelogram along the diagonal AC. Thus, the parallelogram has been divided into two parts and each part is a triangle.

In other words, you get two triangles, ΔABC and ΔADC. Now place ΔADC on ΔABC in such a way that the vertex D falls on the vertex B and the side CD falls along the side AB.

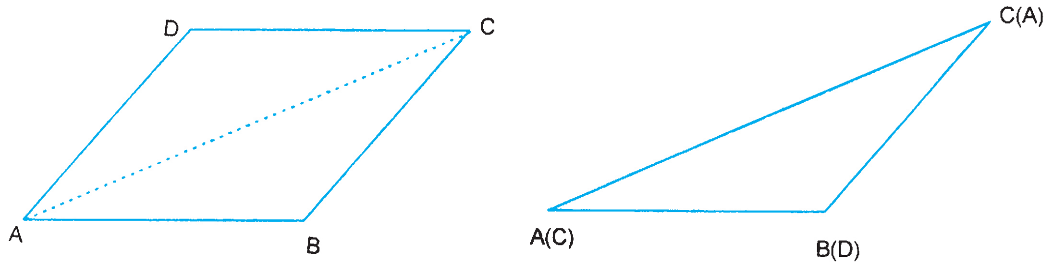

Fig. 13.13

Where does the point C fall?

Where does the point A fall?

You will observe that ΔADC will coincide with ΔABC. In other words $\Delta ABC \cong \Delta ADC$. Also $AB = CD$ and $BC = AD$ and $\angle B = \angle D$.

You may repeat this activity by taking some other parallelograms, you will always get the same results as verified earlier, thus, proving the above two properties of the parallelogram.

Now you can prove the third property of the parallelogram, i.e., the diagonals of a parallelogram bisect each other.

Again take a thin cardboard. Draw any parallelogram PQRS on it. Draw its diagonals

PR and QS which intersect each other at O as shown in Fig. 13.14. Now cut the parallelogram PQRS.

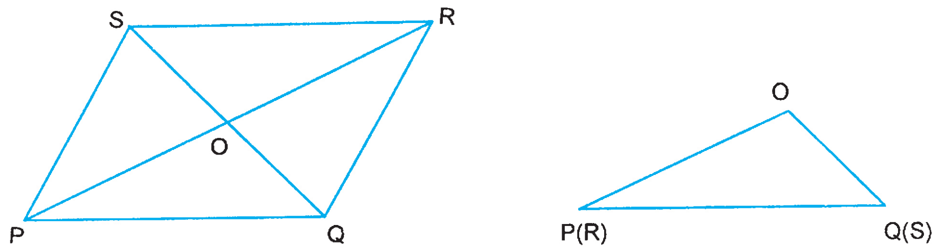

Fig. 13.14

Also cut ΔPOQ and ΔROS.

Now place ΔROS and ΔPOQ in such a way that the vertex R coincides with the vertex P and RO coincides with the side PO.

Where does the point S fall?

Where does the side OS fall?

Is $\Delta ROS \cong \Delta POQ$? Yes, it is.

So, what do you observe?

We find that RO = PO and OS = OQ

You may also verify this property by taking another pair of triangles i.e. ΔPOS and ΔROQ You will again arrive at the same result.

You may also verify the following properties which are the converse of the properties of a parallelogram verified earlier.

A quadrilateral is a parallelogram if its opposite sides are equal.

A quadrilateral is a parallelogram if its opposite angles are equal.

A quadrilateral is a parallelogram if its diagonals bisect each other.

2. Properties of a Rhombus

In the previous section we have defined a rhombus. We know that a rhombus is a parallelogram in which a pair of adjacent sides is equal. In Fig. 13.15, ABCD is a rhombus.

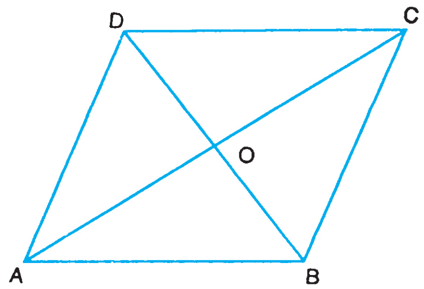

Fig. 13.15

Thus, ABCD is a parallelogram with AB = BC. Since every rhombus is a parallelogram, therefore all the properties of a parallelogram are also true for rhombus, i.e.

(i) Opposite sides are equal,

i.e., AB = DC and AD = BC

(ii) Opposite angles are equal,

i.e., $\angle$ A = $\angle$ C and $\angle$ B = $\angle$ D

(iii) Diagonals bisect each other

i.e., AO = OC and DO = OB

Since adjacent sides of a rhombus are equal and by the property of a parallelogram opposite sides are equal. Therefore,

AB = BC = CD = DA

Thus, all the sides of a rhombus are equal. Measure ∠AOD and ∠BOC.

What is the measures of these angles?

You will find that each of them equals 90°

Also $\angle AOB = \angle COD$ (Each pair is a vertically opposite angles)

and $\angle BOC = \angle DOA$

$\therefore \angle AOB = \angle COD = \angle BOC = \angle DOA = 90°$

Thus, the diagonals of a rhombus bisect each other at right angles.

You may repeat this experiment by taking different rhombuses, you will find in each case, the diagonals of a rhombus bisect each other.

Thus, we have the following properties of a rhombus.

All sides of a rhombus are equal

Opposite angles of a rhombus are equal

The diagonals of a rhombus bisect each other at right angles.

3. Properties of a Rectangle

We know that a rectangle is a parallelogram one of whose angles is a right angle. Can you say whether a rectangle possesses all the properties of a parallelogram or not?

Yes it possesses. Let us study some more properties of a rectangle.

Draw a parallelogram ABCD in which $\angle B = 90°$.

Join AC and BD as shown in the Fig. 13.16

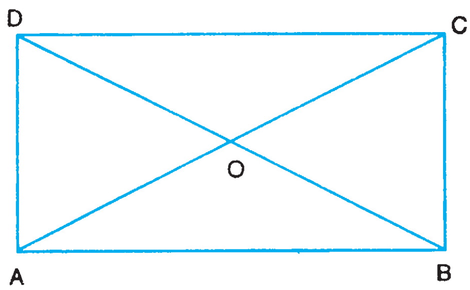

Fig. 13.16

Measure ∠BAD, ∠BCD and ∠ADC, what do you find?

What are the measures of these angles?

The measure of each angle is 90°. Thus, we can conclude that

$$\angle A = \angle B = \angle C = \angle D = 90^\circ$$

i.e., each angle of a rectangle measures 90°. Now measure the diagonals AC and BD. Do you find that AC = BD.

Now, measure AO, OC, BO and OD.

You will find that AO = OC and BO = OD.

Draw some more rectangles of different dimensions. Label them again by ABCD. Join AC and BD in each case. Let them intersect each other at O. Also measure AO, OC and BO, OD for each rectangle. In each case you will find that

The diagonals of a rectangle are equal and they bisect each other. Thus, we have the following properties of a rectangle;

The opposite sides of a rectangle are equal

Each angle of a rectangle is a right-angle.

The diagonals of a rectangle are equal.

The diagonals of a rectangle bisect each other.

4. Properties of a Square

You know that a square is a rectangle, with a pair of adjacent sides equal. Now, can you conclude from definition of a square that a square is a rectangle and possesses all the properties of a rectangle? Yes it is. Let us now study some more properties of a square.

Draw a square ABCD as shown in Fig. 13.17.

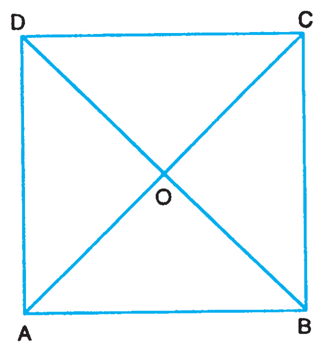

Fig 13.17

Since ABCD is a rectangle, therefore we have

(i) AB = DC, AD = BC

(ii) $\angle A = \angle B = \angle C = \angle D = 90^o$

(iii) AC = BD and AO = OC, BO = OD

But in a square we have AB = AD

∴ By property (i) we have

AB = AD = CD = BC.

Since a square is also a rhombus. Therefore, we conclude that the diagonals AC and BD of a square bisect each other at right angles.

Thus, we have the following properties of a square.

All the sides of a square are equal

Each of the angles measures 90º.

The diagonals of a square are equal.

The diagonals of a square bisect each other at right angles.

Let us study some examples to illtustrate the above properties:

Example 13.3: In Fig. 13.17, ABCD is a parallelogram. If $\angle A = 80^o$, find the measures of the remaining angles

Solution: As ABCD is a parallelogram.

$\angle A = \angle C$ and $\angle B = \angle D$

It is given that

$\angle A = 80^o$

∴ $\angle C = 80^o$

∴ AB || DC

∴ $\angle A + \angle D = 180^o$

∴ $\angle D = (180 - 80)^o = 100^o$

∴ $\angle B = \angle D = 100^o$

Hence $\angle C = 80^o$, $\angle B = 100^o$ and $\angle D$ 100^o

Fig 13.18

Example 13.4: Two adjacent angles of a rhombus are in the ratio 4 : 5. Find the measure of all its angles.

Solution: Since opposite sides of a rhombus are parallel, the sum of two adjacent angles of a rhombus is 180º.

Let the measures of two angles be $4x^o$ and $5x^o$,

Therefore, $4x + 5x = 180$

i.e. $9x = 180$

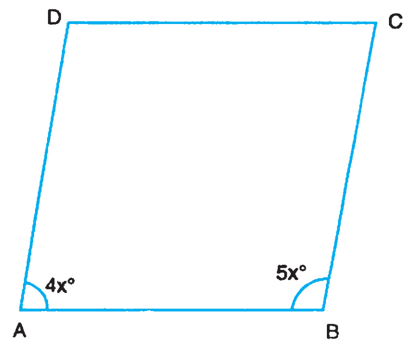

Fig 13.19

$$x = 20$$

$\therefore$ The two measures of angles are 80° and 100°.

i.e. $\angle A = 80^\circ$ and $\angle B = 100^\circ$

Since $\angle A = \angle C \Rightarrow \angle C = 100^\circ$

Also, $\angle B = \angle D \Rightarrow \angle D = 100^\circ$

Hence, the measures of angles of the rhombus are 80°, 100°, 80° and 100°.

Example 13.5: One of the diagonals of a rhombus is equal to one of its sides. Find the angles of the rhombus.

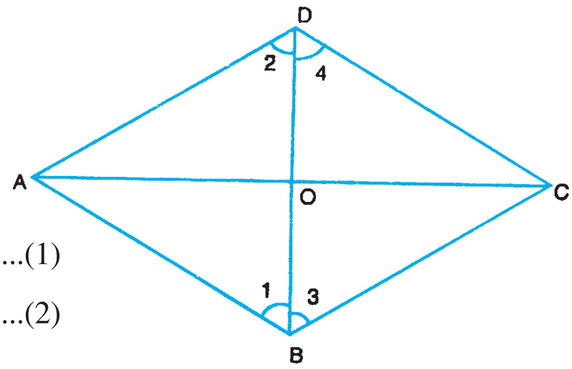

Fig 13.20

Solution: Let in rhombus, ABCD,

$$AB = AD = BD$$

$\therefore$ ΔABD is an equilateral triangle.

$\therefore$ $\angle DAB = \angle 1 = \angle 2 = 60^\circ$ ....(1)

Similarly $\angle BCD = \angle 3 = \angle 4 = 60^\circ$ ....(2)

Also from (1) and (2)

$$\angle ABC = \angle B = \angle 1 + \angle 3 = 60^\circ + 60^\circ = 120^\circ$$

$$\angle ADC = \angle D = \angle 2 + \angle 4 = 60^\circ + 60^\circ = 120^\circ$$

Hence, $\angle A = 60^\circ$, $\angle B = 120^\circ$, $\angle C = 60^\circ$ and $\angle D = 120^\circ$

Example 13.6: The diagonals of a rhombus ABCD intersect at O. If $\angle ADC = 120^\circ$ and OD = 6 cm, find

(a) $\angle OAD$

(b) side AB

(c) perimeter of the rhombus ABCD

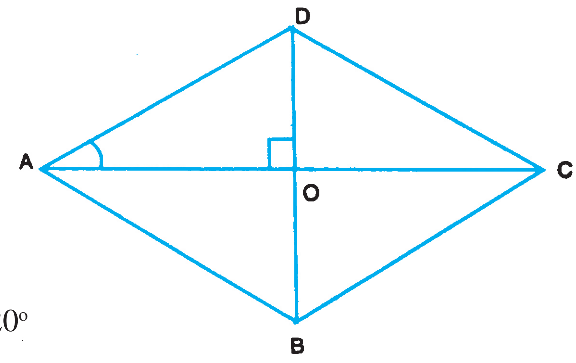

Fig 13.21

Solution: (a) Given that

$$\angle ADC = 120^\circ$$

i.e. $\angle ADO + \angle ODC = 120^\circ$

But $\angle ADO = \angle ODC$ $(\Delta AOD \cong \Delta COD)$

$\therefore$ $2\angle ADO = 120^\circ$

i.e. $\angle ADO = 60^\circ$ ...(i)

Also, we know that the diagonals of a rhombus bisect each that at 90°.

$\therefore \quad \angle DOA = 90^\circ \quad ...(ii)$

Now, in ΔDOA

$$\angle ADO + \angle DOA + \angle OAD = 180^\circ$$

From (i) and (ii), we have

$$60^\circ + 90^\circ + \angle OAD = 180^\circ$$

$$\Rightarrow \quad \angle OAD = 30^\circ$$

(b) Now, $\angle DAB = 60^\circ$ [since $\angle OAD = 30^\circ$, similarly $\angle OAB = 30^\circ$]

$\therefore \Delta DAB$ is an equilateral triangle.

$$OD = 6 \text{ cm} \qquad \text{[given]}$$

$$\Rightarrow \quad OD + OB = BD$$

$$6 \text{ cm} + 6 \text{ cm} = BD$$

$$\Rightarrow \quad BD = 12 \text{ cm}$$

so, $AB = BD = AD = 12$ cm

$$AB = 12 \text{ cm}$$

(c) Now Perimeter $= 4 \times$ side

$= (4 \times 12)$ cm

$= 48$ cm

Hence, the perimeter of the rhombus = 48 cm.

CHECK YOUR PROGRESS 13.2

1. In a parallelogram ABCD, $\angle A = 62^\circ$. Fing the measures of the other angles.
2. The sum of the two opposite angles of a parallelogram is 150°. Find all the angles of the parallelogram.
3. In a parallelogram ABCD, $\angle A = (2x + 10)^\circ$ and $\angle C = (3x - 20)^\circ$. Find the value of x.
4. ABCD is a parallelogram in which $\angle DAB = 70^\circ$ and $\angle CBD = 55^\circ$. Find $\angle CDB$ and $\angle ADB$.
5. ABCD is a rhombus in which $\angle ABC = 58^\circ$. Find the measure of $\angle ACD$.

6. In Fig. 13.22, the diagonals of a rectangle PQRS intersect each other at O. If $\angle ROQ = 40^\circ$, find the measure of $\angle OPS$.

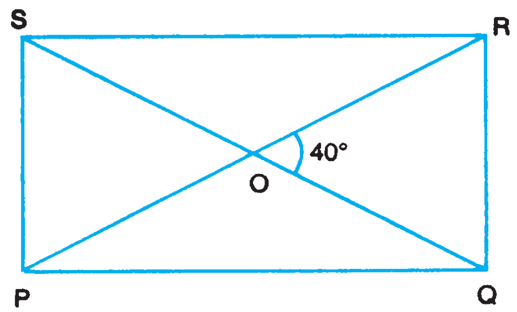

Fig 13.22

7. AC is one diagonal of a square ABCD. Find the measure of $\angle CAB$.

13.4 MID POINT THEOREM

Draw any triangle ABC. Find the mid points of side AB and AC. Mark them as D and E respectively. Join DE, as shown in Fig. 13.23.

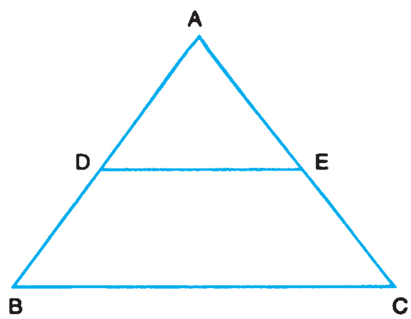

Fig 13.23

Measure BC and DE.

What relation do you find between the length of BC and DE?

Of course, it is $DE = \frac{1}{2} BC$

Again, measure $\angle ADE$ and $\angle ABC$.

Are these angles equal?

Yes, they are equal. You know that these angles make a pair of corresponding angles. You know that when a pair of corresponding angles are equal, the lines are parallel

$\therefore \quad DE \parallel BC$

You may repeat this expreiment with another two or three triangles and naming each of them as triangle ABC and the mid point as D and E of sides AB and AC respectively.

You will always find that $DE = \frac{1}{2} BC$ and $DE \parallel BC$.

Thus, we conclude that

In a triangle the line-segment joining the mid points of any two sides is parallel to the third side and is half of it.

We can also verify the converse of the above stated result.

Draw any ΔPQR. Find the mid point of side RQ, and mark it as L. From L, draw a line LX || PQ, which intersects, PR at M.

Measure PM and MR. Are they equal? Yes, they are equal.

You may repeat with different triangles and by naming each of them as PQR and taking each time L as the mid-point of RQ and drawing a line LM || PQ, you will find in each case that RM = MP. Thus, we conclude that

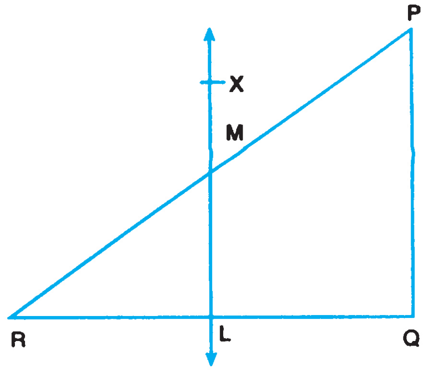

Fig 13.24

"The line drawn through the mid point of one side of a triangle parallel to the another side bisects the third side."

Example 13.7: In Fig. 13.25, D is the mid-point of the side AB of ΔABC and DE || BC. If AC = 8 cm, find AE.

Solution: In ΔABC, DE || BC and D is the mid point of AB

∴ E is also the mid point of AC

$$\text{i.e. } AE = \frac{1}{2} AC$$

$$= \left(\frac{1}{2}\times 8\right) \text{cm} \qquad [\because AC = 8 \text{ cm}]$$

$$= 4 \text{ cm}$$

Hence, $AE = 4$ cm

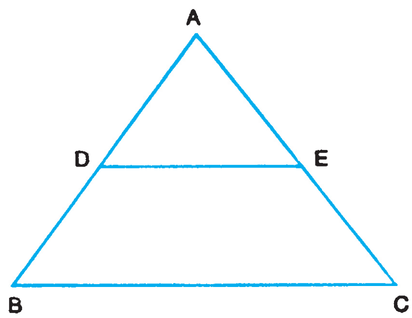

Fig 13.25

Example 13.8: In Fig. 13.26, ABCD is a trapezium in which AD and BC are its non-parallel sides and E is the mid-point of AD. EF || AB. Show that F is the mid-point of BC.

Solution: Since EG || AB and E is the mid-point of AD (considering ΔABD)

∴ G is the mid point of DB

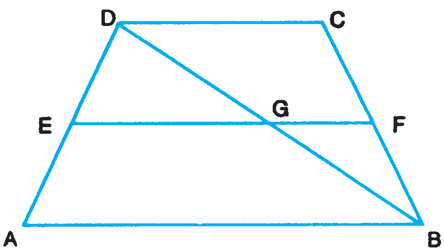

Fig 13.26

In ΔDBC, GF || DC and G is the mid-point of DB,

∴ F is the mid-point of BC.

Example 13.9: ABC is a triangle, in which P, Q and R are mid-points of the sides AB, BC and CA respectively. If AB = 8 cm, BC = 7 cm and CA = 6 cm, find the sides of the triangle PQR.

Solution: P is the mid-point of AB and R the mid-point of AC.

∴ PR || BC and PR $= \frac{1}{2}$ BC

$= \frac{1}{2} \times 7$ cm [∵ BC = cı

= 3.5 cm

Similarly, PQ $= \frac{1}{2}$ AC

$= \frac{1}{2} \times 6$ cm [∵ AC = 6 cm]

= 3 cm

and QR $= \frac{1}{2}$ AB

$= \frac{1}{2} \times 8$ cm [∵ AB = 8 cm]

= 4 cm

Hence, the sides of ΔPQR are PQ = 3 cm, QR = 4 cm and PR = 3.5 cm.

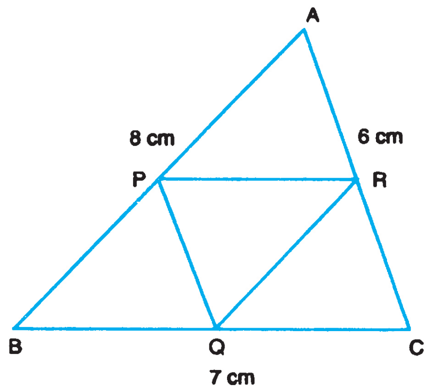

Fig 13.27

CHECK YOUR PROGRESS 13.3

1. In Fig. 13.28, ABC is an equilateral triangle. D, E and F are the mid-points of the sides AB, BC and CA respectively. Prove that DEF is also an equilateral triangle.

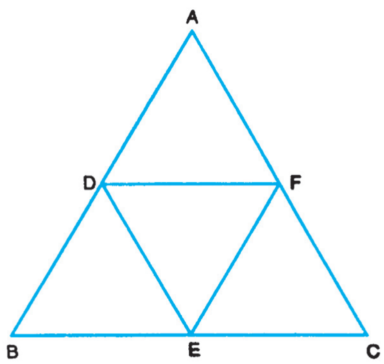

Fig. 13.28

2. In Fig. 13.29, D and E are the mid-points of the sides AB and AC respectively of a ΔABC. If BC = 10 cm; find DE.

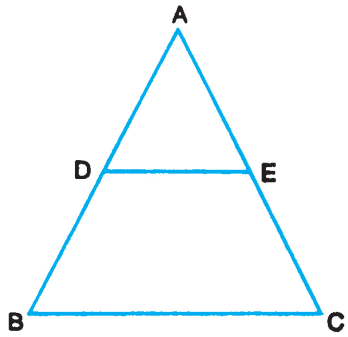

Fig. 13.29

3. In Fig. 13.30, AD is a median of the ΔABC and E is the mid-point of AD, BE is produced to meet AC at F. DG || EF, meets AC at G. If AC = 9 cm, find AF.

   [Hint: First consider ΔADG and next consider ΔCBF]

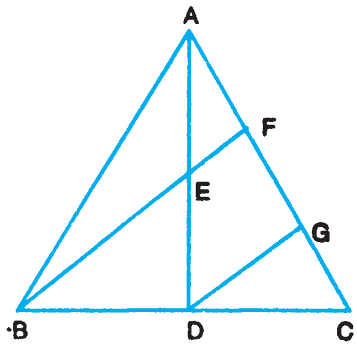

Fig. 13.30

4. In Fig. 13.31, A and C divide the side PQ of ΔPQR into three equal parts, AB||CD||QR. Prove that B and D also divide PR into three equal parts.

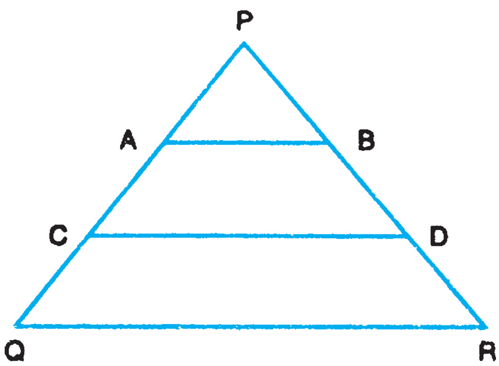

Fig. 13.31

5. In Fig. 13.32, ABC is an isosceles triangle in which AB = AC. M is the mid-point of AB and MN||BC. Show that ΔAMN is also an isosceles triangle.

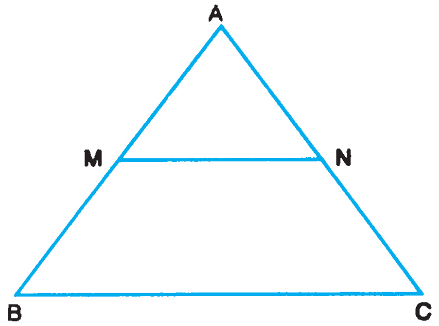

Fig. 13.32

13.5 EQUAL INTERCEPT THEORM

Recall that a line which intersects two or more lines is called a transversal. The line-segment cut off from the transversal by a pair of lines is called an intercept. Thus, in Fig. 13.33, XY is an intercept made by line *l* and m on transversal n.

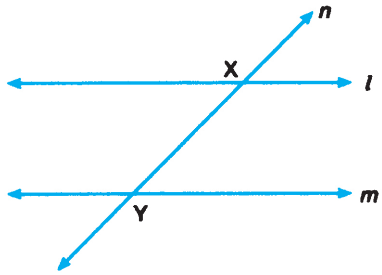

Fig. 13.33

The intercepts made by parallel lines on a transversal have some special properties which we shall study here.

Let *l* and m be two parallel lines and XY be an intercept made on the transversal "n". If there are three parallel lines and they are intersected by a transversal, there will be two intercepts AB and BC as shown in Fig. 13.34 (ii).

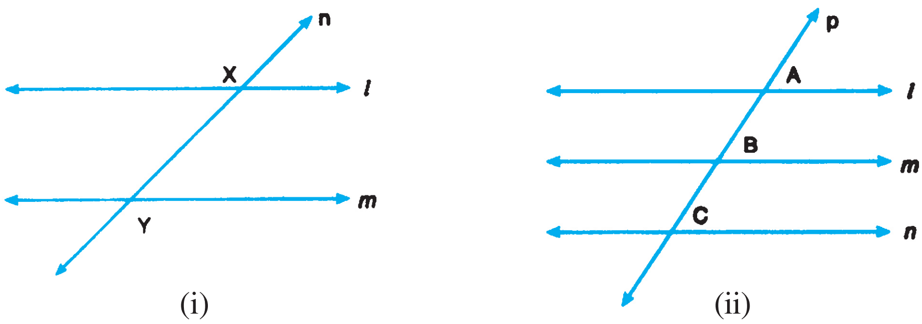

Fig. 13.34

Now let us learn an important property of intercepts made on the transversals by the parallel lines.

On a page of your note-book, draw any two transversals *l* and m intersecting the equidistant parallel lines p, q, r and s as shown in Fig. 13.35. These transversals make different intercepts. Measure the intercept AB, BC and CD. Are they equal? Yes, they are equal.

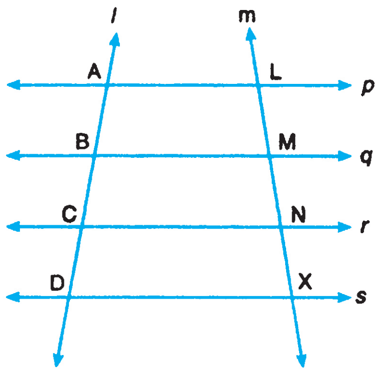

Fig. 13.35

Also, measure LM, MN and NX. Do you find that they are also equal? Yes, they are.

Repeat this experiment by taking another set of two or more equidistant parallel lines and measure their intercepts as done earlier. You will find in each case that the intercepts made are equal.

Thus, we conclude the following:

If there are three or more parallel lines and the intercepts made by them on a transversal are equal, the corresponding intercepts made on any other transversal are also equal.

Let us illustrate it by some examples: This result is known as Equal Intercept Theorm.

Example 13.10: In Fig. 13.36, p || q ||r. The transversal *l*, m and n cut them at L, M, N; A, B, C and X, Y, Z respectively such that XY = YZ. Show that AB = BC and LM = MN.

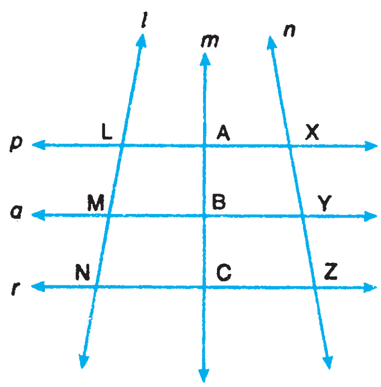

Fig. 13.36

Solution: Given that XY = YZ

∴ AB = BC (Equal Intercept theorem)

and LM = MN

Thus, the other pairs of equal intercepts are

AB = BC and LM = MN.

Example 13.11: In Fig. 13.37, *l* || m || n and PQ = QR. If XZ = 20 cm, find YZ.

Solution: We have PQ = QR

$\therefore$ By intercept theorem,

$$XY = YZ$$

Also $XZ = XY + YZ$

$= YZ + YZ$

$\therefore$ $20 = 2YZ \Rightarrow YZ = 10$ cm

Hence, YZ = 10 cm

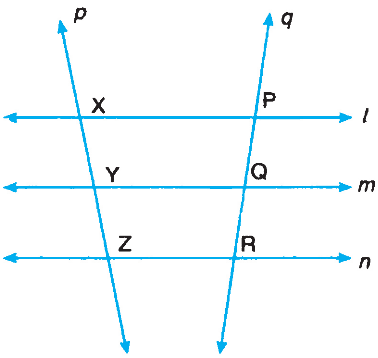

Fig. 13.37

CHECK YOUR PROGRESS 13.4

1. In Fig. 13.38, *l*, m and n are three equidistant parallel lines. AD, PQ and GH are three transversal, If BC = 2 cm and LM = 2.5 cm and AD || PQ, find MS and MN.

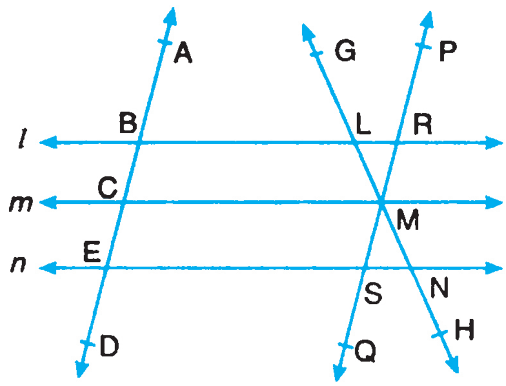

Fig. 13.38

2. From Fig. 13.39, when can you say that AB = BC and XY = YZ?

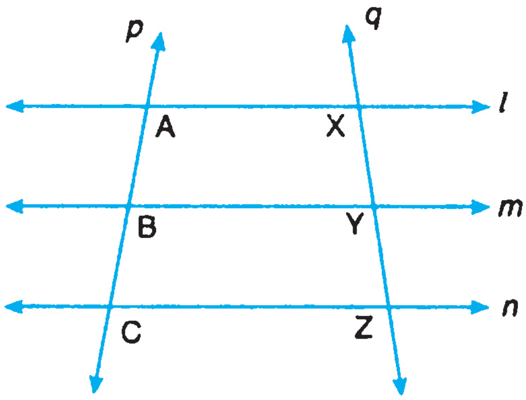

Fig. 13.39

3. In Fig. 13.40, LM = MZ = 3 cm, find XY, XP and BZ. Given that $l \parallel m \parallel n$ and PQ = 3.2 cm, AB = 3.5 cm and YZ = 3.4 cm.

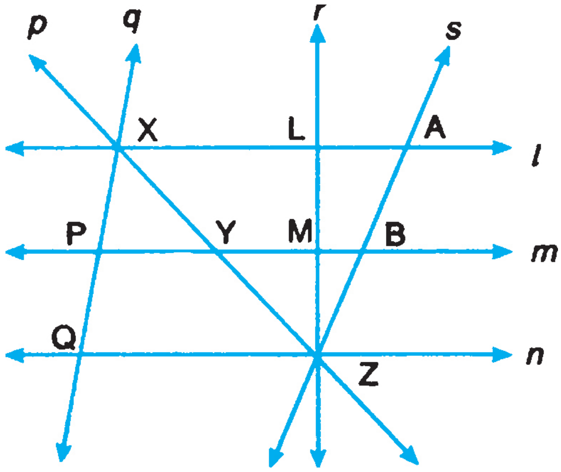

Fig. 13.40

13.6 THE DIAGONAL OF A PARALLELOGRAM AND RELATION TO THE AREA

Draw a parallelogram ABCD. Join its diagonal AC. DP $\perp$ DC and QC $\perp$ DC.

Consider the two triangles ADC and ACB in which the parallelogram ABCD has been divided by the diagonal AC. Because AB $\parallel$ DC, therefore PD = QC.

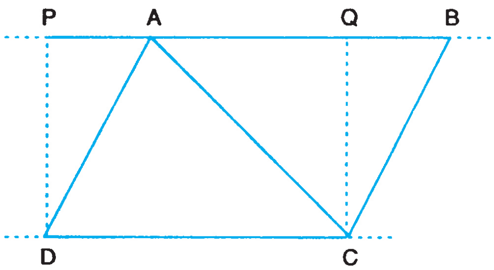

Fig. 13.41

Now, Area of $\Delta ADC = \frac{1}{2}$ DC $\times$ PD ....(i)

Area of $\Delta ACB = \frac{1}{2}$ AB $\times$ QC ....(ii)

As AB = DC and PD = QC

$\therefore$ Area (ΔADC) = Area (ΔACB)

Thus, we conclude the following:

A diagonal of a parallelogram divides it into two triangles of equal area.

13.7 PARALLELOGRAMS AND TRIANGLES BETWEEN THE SAME PARALLELS

Two parallelograms or triangles, having same or equal bases and having their other vertices on a line parallel to their bases, are said to be on the same or equal bases and between the same parallels.

We will prove an important theorem on parallelogram and their area.

Theorm: Parallelogrm on the same base (or equal bases) and between the same parallels are equal in area.

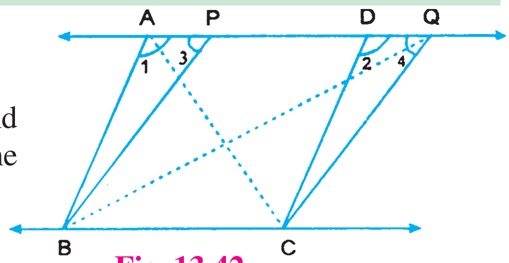

Fig. 13.42

Let us prove it logically.

Given: Parallelograms ABCD and PBCQ stand on the same base BC and between the same parallels BC and AQ.

To prove: Area (ABCD) = Area (BCQP)

we have AB = DC (Opposite sides of a parallelogram)

and BP = CQ (Opposite sides of a parallelogram)

$\angle 1 = \angle 2$

$\therefore \quad \Delta ABP \cong \Delta DCQ$

$\therefore$ Area (ΔABP) = Area (ΔDCQ) ...(i)

Now, Area (||^{gm} ABCD) = Area (ΔABP) + Area Trapezium, BCDP) ...(ii)

Area (||^{gm} BCQP) = Area (ΔDCQ) + Area Trapezium, BCDP) ...(iii)

From (i), (ii) and (iii), we get

Area (||^{gm} ABCD) = Area (||^{gm} BCQP)

Parallelogram on the same base (or equal bases) and between the same parallels are equal in area.

Note: ||^{gm} stands for parallelogram.

Result: Triangles, on the same base and between the same parallels, are equal in area.

Consider Fig. 13.42. Join the diagonals BQ and AC of the two parallelograms BCQP and ABCD respectively. We know that a diagonals of a ||^{gm} divides it in two triangles of equal area.

$\therefore$ Area (ΔBCQ) = Area (ΔPBQ) [Each half of $||^{gm}$ BCQP]

and Area (ΔABC) = Area (ΔCAD) [Each half of $||^{gm}$ ABCD]

$\therefore$ Area (ΔABC) = Area (ΔBCQ) [Since area of $||^{gm}$ ABCD = Area of $||^{gm}$ BCQP]

Thus we conclude the following:

Triangles on the same base (or equal bases) and between the same parallels are equal in area.

13.8 TRIANGLES ON THE SAME OR EQUAL BASES HAVING EQUAL AREAS HAVE THEIR CORRESPONDING ALTITUDES EQUAL

Recall that the area of triangle = $\frac{1}{2}$ (Base) $\times$ Altitude

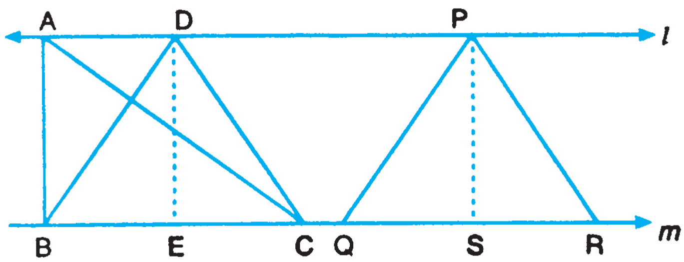

Fig. 13.43

Here BC = QR

and Area (ΔABC) = Area (ΔDBC) = Area (ΔPQR) [Given] ..(i)

Draw perpendiculars DE and PS from D and P to the line m meeting it in E and S respectively.

Now Area (ΔABC) = $\frac{1}{2}$ BC $\times$ DE

Area (ΔDBC) = $\frac{1}{2}$ BC $\times$ DE ...(ii)

and Area (ΔPQR) = $\frac{1}{2}$ QR $\times$ PS

Also, BC = QR (given) ...(iii)

From (i), (ii) and (iii), we get

$$\frac{1}{2} BC \times DE = \frac{1}{2} QR \times PS$$

or $$\frac{1}{2} BC \times DE = \frac{1}{2} BC \times PS$$

$$\therefore \quad DE = PS$$

i.e., Altitudes of ΔABC, ΔDBC and ΔPQR are equal in length.

Thus, we conclude the following:

Triangles on the same or equal bases, having equal areas have their corresponding altitudes equal.

Let us consider some examples:

Example 13.12: In Fig. 13.44, the area of parallelogram ABCD is 40 sq cm. If BC = 8 cm, find the altitude of parallelogram BCEF.

Solution: Area of $\|^{gm}$ BCEF = Area of $\|^{gm}$ ABCD = 40 sq cm

we know that Area ($\|^{gm}$ BCEF) = EF × Altitude

or 40 = BC × Altitude of $\|^{gm}$ BCEF

or 40 = BC × Altitude of $\|^{gm}$ BCEF

$\therefore$ Altitude of $\|^{gm}$ BCEF = $\frac{40}{8}$ cm or 5 cm.

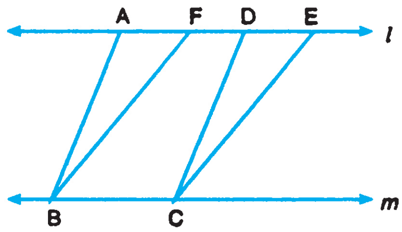

Fig. 13.44

Example 13.13: In Fig. 13.45, the area of ΔABC is given to be 18 cm^2. If the altitude DL equals 4.5 cm, find the base of the ΔBCD.

Solution: Area (ΔBCD) = Area (ΔABC) = 18 cm^2

Let the base of ΔBCD be x cm

$$\therefore \quad \text{Area of } \Delta BCD = \frac{1}{2} x \times DL$$

$$= \left(\frac{1}{2} x \times 4.5\right) cm^2$$

or $$18 = \left(\frac{9}{4} x\right)$$

$$\therefore \quad x = \left(18 \times \frac{4}{9}\right) cm = 8 \text{ cm}.$$

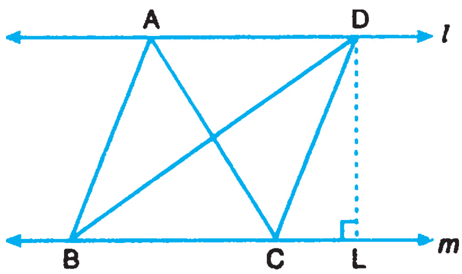

Fig. 13.45

Example 13.14: In Fig. 13.46, ABCD and ACED are two parallelograms. If area of ΔABC equals 12 cm^2, and the length of CE and BC are equal, find the area of the trapezium ABED.

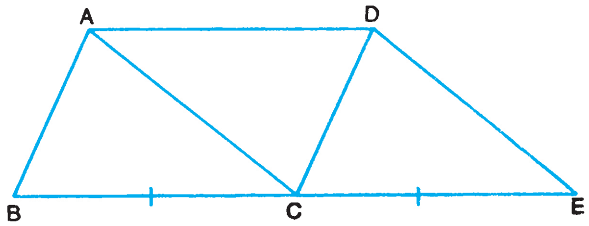

Fig. 13.46

Solution: Area (||^{gm} ABCD) = Area (||^{gm} ACED)

The diagonal AC divides the ||^{gm} ABCD into two triangles of equal area.

$\therefore$ Area (ΔBCD) = $\frac{1}{2}$ Area (||^{gm} ABCD)

$\therefore$ Area (||^{gm} ABCD) = Area (||^{gm} ACED) = 2×12 cm^2

= 24 cm^2

$\therefore$ Area of Trapezium ABED

= Area (ΔABC) + Area (||^{gm} ACED)

= (12 + 24) cm^2

= 36 cm^2

CHECK YOUR PROGRESS 13.5

1. When do two parallelograms on the same base (or equal bases) have equal areas?
2. The area of the triangle ABC formed by joining the diagonal AC of a ||^{gm} ABCD is 16 cm^2. Find the area of the ||^{gm} ABCD.
3. The area of ΔACD in Fig. 13.47 is 8 cm^2. If EF = 4 cm, find the altitude of ||^{gm} BCFE.

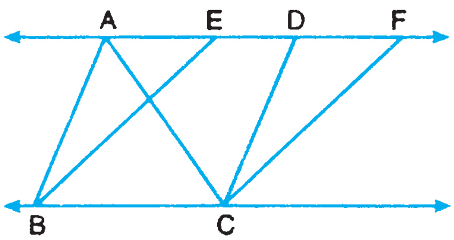

Fig. 13.47

LET US SUM UP

- A quadrilateral is a four sided closed figure, enclosing some region of the plane.
- The sum of the interior or exterior angles of a quadrilateral is equal to 360° each.
- A quadrilateral is a trapezium if its only one pair of opposite sides is parallel.
- A quadrilateral is a parallelogrm if both pairs of sides are parallel.
- In a parallelogram:

  (i) opposite sides and angles are equal.

  (ii) diagonals bisect each other.
- A parallelogram is a rhombus if its adjacent sides are equal.
- The diagonals of a rhombus bisect each other at right angle.
- A parallelogram is a rectangle if its one angle is 90°.
- The diagonals of a rectangle are equal.
- A rectangle is a square if its adjacent sides are equal.
- The diagonals of a square intersect at right angles.
- The diagonal of a parallelogram divides it into two triangles of equal area.
- Parallelogram on the same base (or equal bases) and between the same parallels are equal in area.
- The triangles on the same base (or equal bases) and between the same parallels are equal in area.
- Triangles on same base (or equal bases) having equal areas have their corrsponding altitudes equal.

TERMINAL EXERCISE

1. Which of the following are trapeziums?

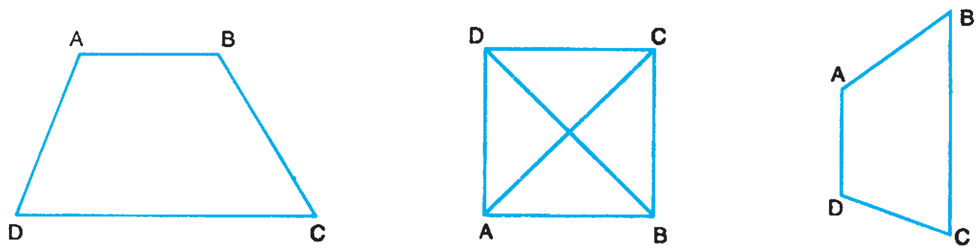

Fig. 13.48

2. In Fig. 13.49, PQ || FG || DE || BC. Name all the trapeziums in the figure.

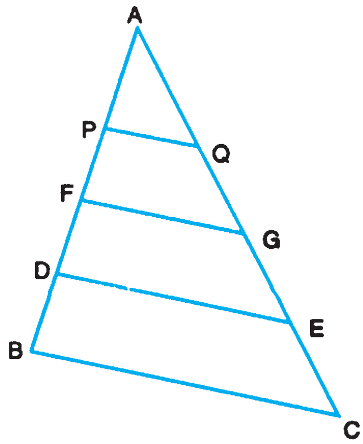

Fig. 13.49

3. In Fig. 13.50, ABCD is a parallelogram with an area of 48 cm^2. Find the area of (i) shaded region (ii) unshaded region.

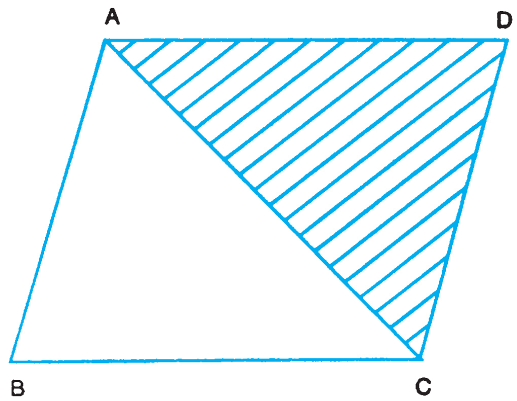

Fig. 13.49

4. Fill in the blanks in each of the following to make them true statements:

   (i) A quadrilateral is a trapezium if ....

   (ii) A quadrilateral is a parallelogram if ....

   (iii) A rectangle is a square if ...

   (iv) the diagonals of a quadrilateral bisect each other at right angle. If none of the angles of the quadrilateral is a right angle, it is a ...

   (v) The sum of the exterior angles of a quadrilateral is ...

5. If the angles of a quadrilateral are $(x-20)^o$, $(x+20)^o$, $(x-15)^o$ and $(x+15)^o$, find x and the angles of the quadrilateral.

6. The sum of the opposite angles of a parallelograms is 180^o. What type of a parallelogram is it?

7. The area of a ΔABD in Fig. 13.51 is 24 cm^2. If DE = 6 cm, and AB ∥ CD, BD ∥ CE, AE ∥ BC, find

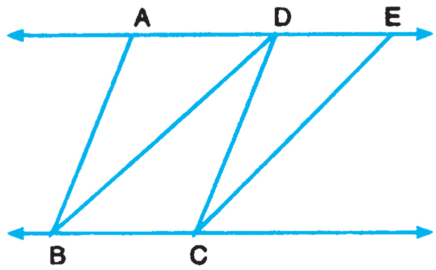

Fig. 13.51

(i) Altitude of the parallelogram BCED.

(ii) Area of the parallelogram BCED.

8. In Fig. 13.52, the area of parallelogram ABCD is 40 cm^2. If EF = 8 cm, find the altitude of ΔDCE.

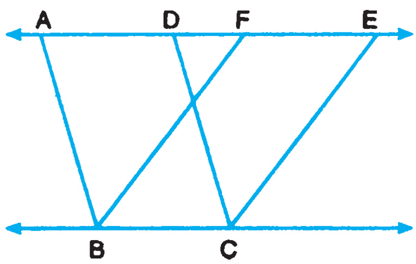

Fig. 13.52

ANSWERS TO CHECK YOUR PROGRESS

13.1

1. (i) Rectangle (ii) Trapezium (iii) Rectangle (iv) Parallelogram
   (v) Rhombus (vi) Square
2. (i) True (ii) False (iii) True (iv) True
   (v) True (vi) True (vii) False (viii) False
   (ix) False (x) False
3. 90°
4. 60°, 84°, 84° and 132°
5. Other pair of opposite angles will also be supplementary.

13.2

1. $\angle B = 118^\circ$, $\angle C = 62^\circ$ and $\angle D = 118^\circ$
2. $\angle A = 105^\circ$, $\angle B = 75^\circ$, $\angle C = 105^\circ$ and $\angle D = 75^\circ$

3. 30
4. $\angle CDB = 55^o$ and $\angle ADB = 55^o$
5. $\angle ACD = 61^o$
6. $\angle OPS = 70^o$ 7. $\angle CAB = 45^o$

13.3

2. 5 cm
3. 3 cm

13.4

1. MS = 2 cm and MN = 2.5 cm
2. 1, m and n are three equidistant parallel lines
3. XY = 3.4 cm, XP = 3.2 cm and BZ = 3.5 cm

13.5

1. When they are lying between the same parallel lines
2. 32 cm^2
3. 4 cm

ANSWERS TO TERMINAL EXERCISE

1. (i) and (iii)
2. PFGQ, FDEG, DBCE, PDEQ, FBCG and PBCQ
3. (i) 24 cm^2 (ii) 24 cm^2
4. (i) any one pair of opposite sides are parallel.
   (ii) both pairs of opposite sids are parallel
   (iii) pair of adjacent sides are equal
   (iv) rhombus
   (v) 360^o
5. $x = 90^o$, angles are 70^o, 110^o, 75^o and 105^o respectively.
6. Rectangle.
7. (i) 8 cm (ii) 48 cm^2
8. 5 cm

You have to grow from the inside out.

none can teach you,
none can make you spiritual.
there is no other teacher
but your own soul''

-SWAMI VIVEKANANDA

For more information please visit QR code

www.bseh.ac.in   contact@bseh.ac.in